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Personal Performance Shapers (PPS): Their Importance in Mathematics Achievement and in Pupils' Empowerment

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Abstract **Original Research Article**

Migrations have increased in the last decades, particularly in Europe, bringing new challenges to schools and societies. In Portugal, despite the Constitution (Assembleia da República, 2005) that claims equal rights for everyone, gender and racial discriminations still exist. César and Machado (2024, submitted, in preparation) illuminated that the mother language (L1) shaped how pupils do and learn Mathematics. Pupils who participate in Cape Verdean culture and whose L1 is Creole prefer a global approach, spatial or geometric reasoning (Battista, 2007), and graphic representation solving strategies, usually not accepted at school - a subtle form of exclusion. Thus, they feel powerless and act as peripheral participants (César, 2009; Lave & Wenger, 1991). César (2009, 2013a, 2013b) showed that collaborative and dialogical work was effective to empower pupils from vulnerable minorities. Using inter-empowerment mechanisms, exploring different mathematical reasoning, and solving strategies, and developing bi-univocal cultural mediation (César, 2017; César & Machado, 2024) promoted pupils' empowerment, and learning. The Interaction and Knowledge (IK) project lasted 12 years and had a 10-year-follow-up. We focus on data from the action-research projects, including 69 Mathematics teacher/researchers, and around 600 classes. The triangulation of sources, data collecting instruments, researchers, and theories were used as quality criteria. Data treatment and analysis were based on a narrative content analysis (Clandinin & Connelly, 1998). Some examples illuminate that changing the first week of classes and using personal performance shapers (PPS) to decide about the first dyads, were essential to empower pupils, and to increase their learning. Implicit messages and inter-empowerment mechanisms also played an important role. Collaborative and dialogical class work and research were important features to increase pupils' empowerment, self-esteem, resilience, and school performances. These results were more striking when schools used bi-univocal cultural mediation.

Keywords: mathematics education, language (L1), collaborative and dialogical work, inter- and intra-empowerment mechanisms, life trajectories of participation, bi-univocal cultural mediation, personal performance shapers (PPS)

INTRODUCTION

Migrations have changed societies and schools in Europe. Since the 70s, immigrants have come to Portugal, particularly from Cape Verde. Now they are second and third generation migrants. But despite being Portuguese, at home they speak Creole, their mother language (L1), in which they think, count, sing, express emotions, and dream. In Cape Verdean culture, singing and dancing are highly valued activities. Children learn them from a very young age. João dos Santos, a Portuguese pedagogue and psychoanalyst, claimed that "dancing, dramatizing is, as we know, at the origin of everything that is thinking" (Carvalho e Branco, 2000, p. 31). Thus, those participating in the culture of Cape Verde develop natural ways of thinking, shaped by their L1 and these cultural activities. César and Machado (2024, submitted, in preparation) show in detail how L1 shapes mathematical performances. Those whose L1 is Creole prefer global approaches to problems, spatial and geometrical reasoning (Battista, 2007), and graphic representation solving strategies, usually not accepted in school classes, or mentioned in textbooks. Those whose L1 is Portuguese prefer step-by-step approaches to problems, analytical reasoning, and arithmetic or algebraic solving strategies. The frustration of being forbidden to use their natural ways of doing and learning Mathematics made many pupils participating in the Cape Verdean culture underachieve in Mathematics and have performances that were far below their abilities and competencies (César, 2009, 2013a, 2013b, 2024; César & Machado, 2024, submitted, in preparation). Feeling powerless and unappreciated had negative

impacts on their life trajectories of participation, before participating in the IK project.

Teachers' awareness about how L1 shapes mathematical performances is a key issue to develop an intercultural and inclusive Mathematics Education, promoting equity and social justice. César (2010) underlines the need to develop teachers' epistemological conscience, enabling them to act as decision makers, when operationalizing the curriculum. Interculturality stresses the need to use bi-univocal cultural mediation, a construct coined by César (2017). When the dominant culture, the one in power, is the only one deciding about cultural mediation, it is a univocal cultural mediation. If the power is distributed between the different cultures, sharing the decisions, operationalization and evaluation of cultural mediation, it is a bi-univocal cultural mediation. It legitimates the participants from those cultures, giving them a voice. Those schools have a polyphony of voices (Bakhtin, 1929/1981) that are listened to, respected, and valued. Diversity is celebrated and used to promote pupils' learning and development.

Mathematics is often rejected by pupils, who show negative social representations associated to it (César, 2009, 2017; César & Machado, submitted). As Marková (2005) claimed, social representations are dialogical. César (2009, 2013a) and César, Machado, and Ventura (2014) illustrated how collaborative and dialogical work promoted more positive social representations, if practices were designed to increase pupils' empowerment and helped them to act as legitimate participants (César, 2009; Lave & Wenger, 1991). Several authors claimed that social interactions played a major role in pupils' mathematical performances, and that cultural issues shaped mathematical performances (César & Kumpulainen, 2009; Cobb & Bauersfeld, 2012; Sfard, 2008; Schubauer-Leoni & Perret-Clermont, 1997). Others went further, assuming a critical Mathematics perspective, underlining the importance of power issues in learning, equity, and social justice (Apple, 1995, 2002; Beccuti & Robutti, 2022; César, 2017; César & Machado, 2024, submitted; Cobb & Hodge, 2007; Reinke, Stephan, & Cobb, 2024; Skovsmose, 2023), illustrating how mathematical practices contribute to an active and critical citizenship. They underlined the importance of implementing mathematical practices that facilitated the transitions among contexts, scenarios, and situations.

César (2009) coined the construct of inter- and intra-empowerment mechanisms. In a similar way to what Vygotsky (1932/1978) claims regarding development, empowerment mechanisms first exist on a social plane (inter-empowerment) and then they are internalized and become individual (intra-empowerment). Inter-empowerment mechanisms allow those in power (e.g., parents, teachers) to distribute that power, facilitating legitimate participation. This is particularly important for pupils who participate in vulnerable and socially undervalued cultures, like the Cape Verdean culture. César (2009, 2017) and César and Machado (submitted) illuminated how important it is to use powerful implicit messages, like the one used in the IK project: everyone can do and learn Mathematics. When teachers use inter-empowerment mechanisms, pupils internalize them, transforming them into intra-empowerment mechanisms they can use in other contexts, scenarios, and situations, in an autonomous way. Using implicit messages and inter-empowerment mechanisms, working in dyads or small groups in a collaborative and dialogical way, using mathematical tasks of different natures, and a coherent evaluation system, were important features of the didactic contract (César, 2009; Schubauer-Leoni, 1986) negotiated with pupils in the IK. Those were the external features that contributed to promote school achievement and empowerment. Succeeding in Mathematics facilitated tracing life trajectories of participation in which long-studies were considered. As Ligorio and César (2013) illustrated, changing practices had strong impacts on the dialogical self of pupils, namely on their I-positions (e.g., as son/daughter, brother/sister, pupil, friend) and on their Me-positions (how the others see me and how I believe they see me). As Hermans (2001, 2003) stated, the dialogical self is a work-in-progress, shaped by social interactions and cultures.

The IK team claimed that knowing pupils' abilities and competencies since the first week of classes was an essential step to empower them and to promote school achievement. We conceive of competencies as knowledge-in-action, that pupils can use in an autonomous way, in other contexts, scenarios, and situations. But we did not want to use psychometric tests, which follow a static perspective of abilities, that are supposed to remain the same for a lifetime (Freeman, 2017). The first intelligence scales used in France (Binet-Simon, 1904) were meant to separate pupils, who would go to regular schools, from those who went to special education schools, promoting segregation and exclusion. The IK team wanted the opposite: to elaborate a developmental evaluation instrument, used by teachers, that would show the mathematical abilities and competencies pupils mobilized, the ones that could be used to construct more knowledge. Vygotsky (1932/1978) states that the abilities pupils already mobilize are part of their real development, while those they can only use when working with a more competent peer, are part of their potential development. Between them, there is the zone of proximal development (ZPD). If pupils work in their ZPD, the potential development becomes real development. César (2009, 2013a, 2013b, 2014) showed that working in dyads and small groups was even better than what Vygotsky expected: both pupils could act alternatively as more competent peer, and they could both progress in their development and learning.

A developmental instrument was elaborated by the IK team: the

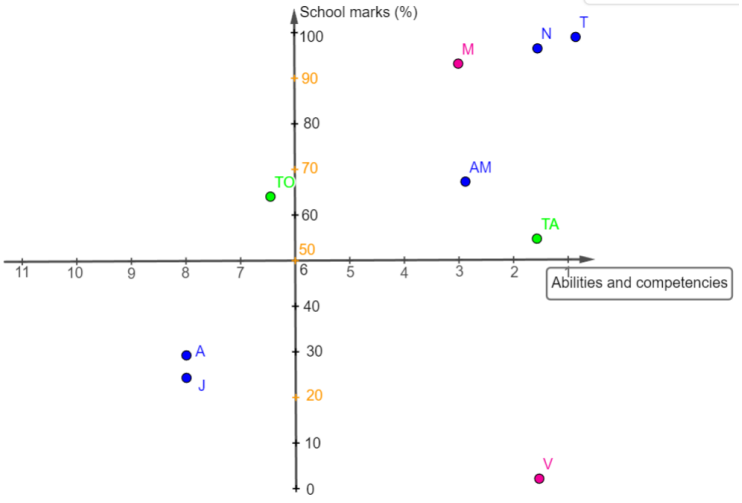
IACC – Instrumento de Avaliação de Capacidades e Competências (César & Machado, 2024, submitted, in preparation; Machado, 2014). The data collected through this instrument proved to be essential to decide about the first dyads, and to promote pupils' empowerment and achievement. In a whole-class discussion, in the next class, each pupil went to the blackboard to explain a successful solving strategy. In the questionnaires, interviews, and informal conversations many pupils stated this was a turning point for them (César, 2009, 2017; César & Machado, submitted).

From the beginning of the IK, we used pupils' abilities and competencies to decide about the first dyads (César, 2009, 2017, 2014; César et al., 2014; César & Machado, 2024, submitted, in preparation). We also considered their school marks. But it was in the meta-analysis of the IK's empirical corpus, that César coined a new concept, discussed for the first time in this paper: Personal Performance Shapers (PPS). Shapers are used in the perspective of the gestalt and Vygotskian (1932/1978) theories. They do not correspond to a neo-behaviourist perspective. The PPS are represented by a referential with two axes, defining four quadrants (Figure 1). On the abscissa axis are the abilities and competencies evaluated by the IACC. On the right part of the axis are the pupils who mobilize abstract reasoning (AR), and on the left those who mobilize concrete reasoning (CR). There are 11 positions: in Position 1 are those who succeeded in the five tasks, and who gave complete explanations; in Position 2 they also succeded in the five tasks, but those explanations were not complete; in Position 3 they mobilized AR (evaluated in Task C) and succeeded in three other tasks; Position 4 is similar with two other tasks; Position 5 with one other task; in Position 6 they only mobilize AR; Position 7 to Position 10 are similar but mobilizing CR. Position 11 is rare. Those pupils need further care, as some of them revealed cognitive issues. On the ordinate axis we considered the school marks of hr ptrvioud school year. The orange percentages identify the different levels: Level 1 (0 to 19%); Level 2 (20 to 49%); Level 3 (50 to 69%); Level 4 (70 to 89%), and Level 5 (90 to 100%). There are two negative (until 49%) and three positive levels. Thus, the intervals between levels are not the same and these two scales are both ordinal ones.

The points in Figure 1 refer to the pupils whose life trajectory of participation was analysed in other papers and chapters. They participated in three different classes: (1) V. and M., in pink, were a dyad discussed by César (2009, 2013a, 2013b, 2024), and César and Machado (2024); (2) Tatiana (TA) and Tomás (TO), in green, were analysed by César (2017) and César and Machado (submitted, in preparation); and (3) J., A., AM., N., and T., in blue, were discussed by César (2014), César and her collaborators (2014), and César and Santos (2006). The first and second classes were from the same school, in the surroundings of Lisbon. The third class was from another school, in Lisbon. Both schools used bi-univocal cultural mediation.

Usually, teachers only consider the school marks from previous school years. Thus, pupils in Quadrants 3 and 4 are described by their teachers as "weak", or "unable to learn Mathematics". Teaxhers' expectations regarding their learning and achievement are usually low, shapping pupils' negative self-esteem, particularly as Mathematics learners. But if we consider the personal performance shapers (PPS), we realize they mobilize a lot of abilities and competencies, sometimes even more than those mobilized by top achievers, as illustrated by V. and M. (César, 2009, 2013a, 2013b, 2024; César and Machado, 2024), or by Tatiana and Tomás (César, 2017; César and Machado, submitted, in preparation). It also allows teachers to know if those abilities and competencies are complementary, the ones that promote more development and learning. In this paper we discuss V. and M.'s dyad, illustrating tin detail these issues and how PPS reveal much more possibilities to promote learning and achievement.

Figure 1 – Personal Performance Shapers (PPS)



J. was in a Special Educational Needs (SEN) condition and was deeply rejected by his colleagues. Reports from teachers and psychologists considered him "unable to learn any Mathematics" (J., teacher' report, 7th grade) and "with a severe cognitive handicap, that also affects his socialization skills" (J., psychologist report, 7th grade). The IACC showed he mobilized concrete reasoning, was able to analyse geometrical figures and use geometrical reasoning, and to make connections between school Mathematics and real-life Mathematics. Thus, he mobilized much more abilities and competencies than those mentioned in his reports. Psychologists used psychometric tests, with a rigid time limit, which penalized him, as he had motor difficulties and talked slower. Thus, using a developmental instrument was an essential step to promote his achievement and socialization. AM. was chosen to be his peer. She was calm, would accept sitting next to him even if she was not pleased (J. was rejected but the whole class), and they could both progress as they had complementary abilities and competencies. César (2014) and César and her colleagues (2014) discuss a small group work among J., A., N., and T., and if anyone did not know that J. had been rejected, s/he would never have imagined that. The two pupils in Quadrant 3 (J. and A.) mobilized less abilities and had lower school marks. But they both progressed a lot. They developed many abilities and competencies, achieved positive marks, and were successful in high school, and in their professional lives illuminating the importance of PPS.

Tatiana (TA) and Tomás (TO) achieved Level 3, but Tomás had a higher percentage, while Tatiana showed better developed abilities and competencies. They mobilized complementary abilities and competencies, and were able to progress while working together and, later, when working with other peers. Thus, progress is consistent in time, an important criterion for the IK team. These examples illustrate how important PPS are to form the first dyads, beginning to empower pupils, and to promote their achievement and socialization, constructing a community of knowledge (Civil, 2007), and developing thinking spaces (Perret-Clermont, 2004), where pupils feel safe to ask about their doubts, questions, and express their natural reasoning and solving strategies. The PPS are key knowledge regarding each class and pupil, allowing teachers to adjust the nature of the mathematical tasks to pupils' abilities, competencies, interests, and needs.

METHOD

The main goals of the *Interaction and Knowledge* (IK) project were: (1) to study and promote collaborative and dialogical work, in class and in research; and (2) to promote an intercultural and inclusive education. The IK lasted 12 years (1994/95-2005/06). It had a 10-year follow-up, in which pupils answered a questionnaire and were interviewed (end of the school year, June/July). Six pupils were interviewed in each class. A meta-analysis of the complete IK's *empirical corpus* took place from 2016 onwards. It allowed a new construct to be coined: bi-univocal cultural mediation (César, 2017). It showed how L1 shapes pupils' performances and Mathematics learning (César & Machado, 2024, submitted, in preparation). In this paper we coin a new concept: Personal Performance Shapers (PPS).

The IK included three research designs: (1) *quasi experimental* studies; (2) action-research projects; and (3) case studies (Hamido & César, 2009). The IK team assumed an interpretative paradigm (Denzin, 2002). This team worked in a collaborative and dialogical way. So, we use "we" even in single authored papers, illuminating the social construction of knowledge. We focus on data from the action-research projects and from Mathematics classes. The main goal of this paper is to show the importance of PPS in the promotion of pupils' empowerment, school achievement and life trajectories of participation.

The participants were: the pupils who participated in the IK project (5th to 12th grade, around 600 classes, from Portugal and Cape Verde); 69 Mathematics teacher/researchers; four psychologists (researchers); pupils' families; the school board and other educational agents; and internal (IK team) and external observers. The diversity of the participants allowed for the triangulation of the sources (informants). Having 73 researchers allowed for the triangulation of the researchers, including their interpretations. These interpretations were also triangulated with pupils whose excerpts we used in the papers or chapters. We used collaborative and dialogical work as a tool for research (César, 2013b; Hamido & César, 2009), promoting an intercultural and inclusive research (César et al., 2014; César & Machado, 2924, submitted). Triangulations are important quality criteria in interpretative research which uses a qualitative data treatment and analysis (Denzin, 2002; Hamido & César 2009).

The first week of classes was meant to know pupils better, particularly their abilities and competencies. Thus, data was collected through a task inspired in projective techniques (TIPT1), a questionnaire (Q1), and the IACC (*Instrumento de Avaliação de Capacidades e Competências*), a developmental instrument to evaluate pupils' abilities and competencies related to Mathematics. The IACC was elaborated by the IK team

(Machado, 2014). The other instruments were questionnaires (Q), tasks inspired in projective techniques (TIPT), interviews (I), informal conversations (IC), observation (O), audio and videotaped, teacher/researchers' diary (D); school reports (R); reports from external (RE) and internal (RI) observers; and documents (DOC). We triangulated the collecting data instruments.

In action-research projects data was collected for a minimum of one school year. Older teacher/researchers were already part of the school board. They could follow the same class during a cycle (e.g., 3rd cycle, i.e., 7th to 9th grade). But if teacher/researchers were trainee teachers or very young, usually they changed school every year. In classes whose pupils showed highly disruptive ways of acting, the reports (R) from previous years were read before the beginning of the school year (September). Other documents were collected before the beginning of the school year (e.g., the school educational project). In the first week of classes, we used the TIPT1, the Q1 and the IACC. The first interview (I1) took place at the end of the first term (mid-December). The Q2 and the TIPT2 were used in the first class of the second term (beginning of January). The TIPT3, Q3 and I2 were used at the end of the school year (June/July). The other collecting instruments were used during the whole school year.

Data treatment and analysis were based on a narrative content analysis (Clandinin & Connelly, 1998), from which inductive categories emerged. It allowed us to trace pupils' life trajectories of participation (César, 2009, 2013a). In the excerpts, using "…" means a small pause, (…) a longer pause, and […] that a part of the talk was cut. To guarantee pupils remained anonymous, we used the first letter of their names (e.g., V. and M.). Later, as there were no more different letters available, we used a fake name, chosen by the pupils, like Tatiana and Tomás (César, 2017; César & Machado, submitted, in preparation).

Pupils gave permission to use any excerpt we chose for a paper. However, if they felt we should not use it, we deleted that part. Thus, they read and discussed with us each interpretation that concerned them. Sometimes they suggested a complementary explanation, a different analysis, or asked us to also use another excerpt (César, 2013b). So, the IK team respects a more demanding notion of informed consent (César, 2017; César & Machado, submitted), which is much more than merely giving authorization to use their data before the beginning of the research.

RESULTS

The school attended by V. and M. was a challenging one (César, 2009, 2013b, 2017). Two thirds of the pupils came from a poor slum, where many houses had no piped water or electricity. Pupils' parents had jobs that were not socially valued, low salaries, and poor working conditions. Fathers had usually completed primary schooling, but many mothers did not go to school. Many people from Cape Verdean culture lived in this slum. There were also some Portuguese people, from very poor families. Although living in the same slum area, these two communities did not communicate. The other third of the pupils came from some new blocks of flats and villas, where people from middle and upper-middle class lived. This school experienced a very high percentage of underachievement, particularly in Mathematics. Thus, from 1994/95 on the IK project worked there. Outside classes, regulatory dynamics of participation were implemented between the school and the families (César, 2013b) and, later, a bi-univocal cultural mediation (César, 2017; César & Machado, 2024, submitted). In Mathematics classes, the IK developed collaborative and dialogical work, in dyads and small groups, followed by a whole-class discussion in which different approaches to problems, reasoning, and solving strategies were discussed (César, 2009, 2013a, 2013b; César & Machado, 2024, submitted, in preparation).

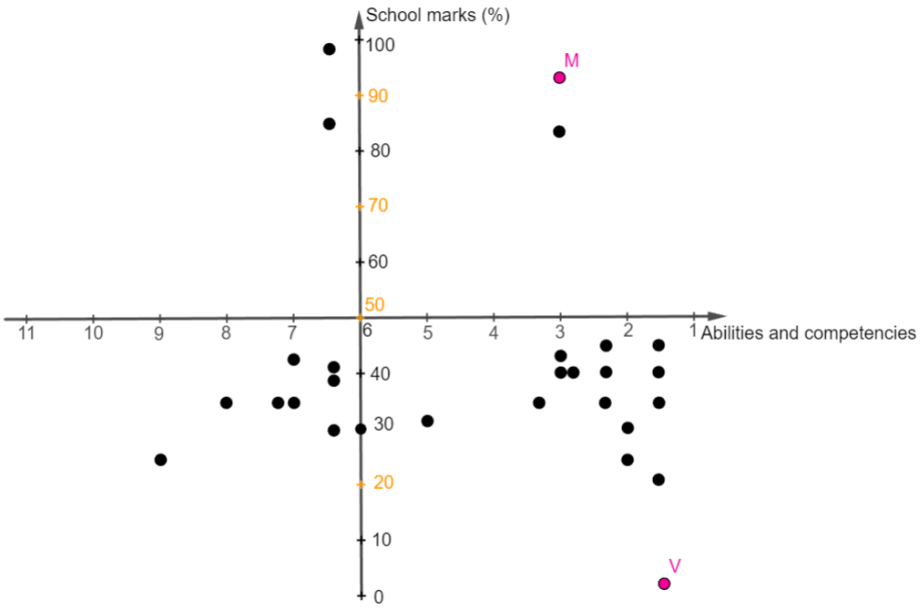
Personal Performance Shapers (PPS): A key concept to promote achievement

Teachers only considered the school marks to identify challenging classes and pupils. The IK project team elaborated a developmental evaluation instrument, the IACC, to know pupils' abilities and competencies from the first week of classes. In that week pupils did not learn new mathematical contents. The TIPT1, Q1, IACC, and observation (O) were used to know pupils better, decide about the first dyads, and the mathematical tasks they would use. When the meta-analysis of the IK's *empirical corpus* took place, the importance of the PPS became clear: using this concept, discussed for the first time in this paper, one could better understand the characterization of each class, and the potentialities of each pupil (Figures 1 and 2).

As teachers only considered the school marks, V. and M.'s class was one they would avoid teaching. This class also had some disciplinary issues. Thus, both teachers and the pupils were quite astonished when a teacher/researcher from the IK team volunteered to teach it, in the 9th grade.

Figure 2 represents the point cloud of the reference frame of V. and M.'s class.

Figure 2 – V. and M.'s class: Personal Performance Shapers (PPS)



It was formed by 28 pupils (9th graders), and 24 of them had a negative mark in Mathematics (Quadrants 3 and 4). The four girls who had Level 4 or 5 (Quadrants 1 and 2) were from the new neighbourhood, and from upper-middle-class families. They were appreciated by teachers and rejected by their classmates. Thus, they had no achievement issues, but experienced socialization difficulties. Figure 2 shows that many pupils had quite developed abilities and competencies, which allowed them to quickly progress if they worked in dyad, with a peer who mobilized complementary abilities and competencies, while solving adequate mathematical tasks, which allowed them to work in their ZPD (César, 2009; César & Machado, 2024, submitted, in preparation; Vygotsky, 1932/1978). Time would prove that: all pupils completed the 9th grade, and they were also able to have a positive mark in the final exam, which was elaborated and corrected by other teachers, who did not believe these pupils could succeed. Most pupils ended high school in a professional course, or went to college (e.g., V. and M.). Thus, for those who experienced underachievement in previous school grades, working in dyads or small groups, in a collaborative and dialogical way, exploring different approaches to problems, reasoning, and solving strategies, promoted their self-esteem, resilience, achievement, and learning. Having access to Mathematics achievement, and improving their school marks in other subjects, made them question the life trajectory of participation they had planed. Thus, even those who thought they would begin working after the 9th grade, chose to go on studying. This illuminates how important are teachers' practices, and how they shape pupils' performances and choices, particularly at school, and in their professional lives, which are also shaped by their abilities and competencies.

Knowing V. and M. better: The importance of the first week of classes

V. was the only pupil who had Level 1. Pupils seldom have this level, because it means they do not solve tasks, answer teachers' questions, or copy the resolutions from the blackboard. V. explained, in the first interview, why this happened, and how his previous experiences in Mathematics classes made him give up trying. Thus, ignoring how L1 shapes pupils' approaches, reasoning, and solving strategies may turn a top pupil into a poor achiever.

> I thought I was helpless. I was too dumb to learn Maths. I always failed, teachers didn't like my way of doing Maths, my need to see it, to draw to understand the problem. When I drew, while solving problems, they were annoyed, they thought I was joking. So, they made many unkind comments. They told me I'd never learn and that I was a lost case. So, I really didn't feel like trying anymore, and I gave up trying. (V., I1, December 1997)

But after answering the TIPT1, where he was asked to draw or write what Mathematics was to him, and the Q1, where he mentioned the subjects he preferred and liked less, his social representation as a Mathematics learner, and his plans to start

working after the 9th grade, V. realized this was a different kind of teacher. No one had ever asked him what he thought about school, or his future. Thus, he looked at the IACC and decided to begin with the tasks that had figures, the ones in which he could use his spatial and geometrical reasoning (Battista, 2007). Then he tried the other ones, and was successful in them all (Position 2), even if he did not give a complete explanation in Task A. Thus, he had very well-developed Mathematical abilities and competencies. His teacher/researcher was convinced that if he worked with M., he could progress a lot, even if he knew they would not be pleased to work together. Time proved their teacher/researcher was right: they became top students, even in high school and college; and they became good friends.

The teacher/researcher's decisions for the first dyads were based on the PPS. He chose four boys to work in dyads with the four girls who had a positive mark. Those boys participated in the Cape Verdean culture and were in Position 2 of the IACC, in Quadrant 4. Their developed abilities and competencies were essential to facilitate their achievement, and they all had a positive mark in the first individual written test (end of October 1997), just five weeks after working in dyads. The four girls also had higher percentages. Thus, in November, he changed dyads. That class already had more pupils with a positive mark, and he needed them to develop new abilities and competencies, and to help other classmates, who still had Level 2. These choices, based on the PPS, transformed a low-performance class into a high-achieving class, and pupils who acted as peripheral participants into legitimated participants (César, 2009, 2013a, 2013b), who felt empowered and expressed their voices.

The IACC showed that V. mobilized critical sense when reading mathematical information, as well as mathematical intuition, abstract reasoning, creativity, connections between school Mathematics and real-life situations, and that he preferred a global approach, spatial or geometrical reasoning, and graphic representation solving strategies. M. had no critical sense, creativity, or mathematical intuition, but mobilized abstract reasoning and made connections between school Mathematics and real-life situations. She preferred a step-by-step approach, analytical reasoning, and arithmetic or algebraic solving strategies. She had more knowledge about Mathematics contents, but she aimed at entering Medicine, and she did not have well-developed spatial and geometric reasoning. The first term of the 10th grade was about Geometry. Thus, she could profit from working with V. if the mathematical tasks used allowed them both, alternately, to act as more competent peer. Thus, choosing the best tasks was one of the challenges for their teacher/researcher.

Task A (Figure 3) was the one in which pupils were less successful. When this instrument was elaborated, different ordering of the five tasks was tried. If Task A was the first one, less pupils rejected the IACC. Thus, this is the first task (Task A). In the whole-class discussion, as teacher/researchers begin with the tasks in which more pupils were successful, they begin with Task C, then Task D or E, followed by Task B, and they finish with Task A (César & Machado, submitted; Machado, 2014). The implicit message is that each pupil can do and learn Mathematics. Motivation of the pupils in Quadrant 3 and 4 is essential and beginning to discuss the tasks that were well done by more pupils is important to achieve that. Thus, each detail of the whole-class discussion of the IACC was carefully planed by the IK team, and teachers participate in in- or pre-service workshops to get ready to use this instrument, and to work in this innovative diactic contract. Teachers' education is a critical issue to achieve the goals of the IK research project.

Task A (Figure 3) describes robberies in an imaginary town (Fractopolis) and includes several mathematical mistakes (e.g., 1 in 20 is not 20%; the robberies did not decrease). Some pupils identify them (e.g., V.), others just repeat what is written in the news (e.g., M.), showing no critical sense regarding mathematical contents in news. As in this paradigmatic dyad, sometimes those who underachieve show more developed abilities and competencies than top achievers.

Figure 3 – Task A – IACC

1. Comment, from the mathematical point of view, the following news:

Decreasing of the Robberies in Homes

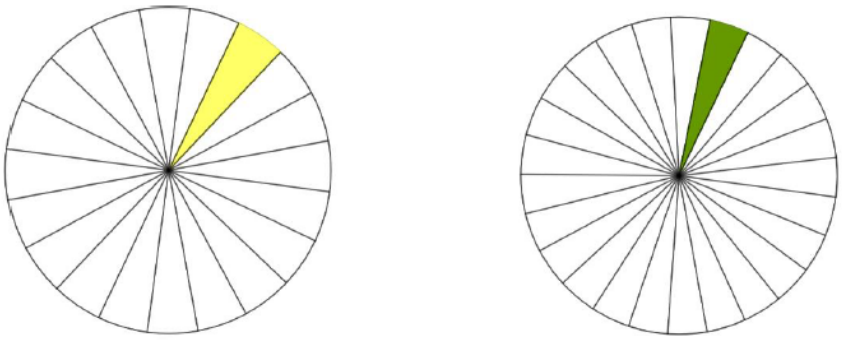
From our correspondent in Fractopolis:

Police told that during last year, in Fractopolis, 1 in each 25 homes had been robbed. This year, the number of robbed homes changed to 1 in 20. So, there was a significant decrease but, even so, 20% is still a very high number. We hope that Fractopolis police will act in the right way to increase the safety of the citizens.

As discussed by César and Machado (2024, submitted, in preparation), L1 shapes the approach, reasoning and solving strategies that pupils prefer. Those whose L1 is Creole prefer a global approach, spatial or geometrical reasoning, and a graphic representation solving strategy, like the ones V. used. The metaphor of the pizza is common, even among those who were

too poor to eat pizzas at home, or to go to a restaurant. This shows they established connections between school Mathematics and real-life situations. It also illustrates their creativity.

Figure 4 – Task A – L1: Creole (V.)



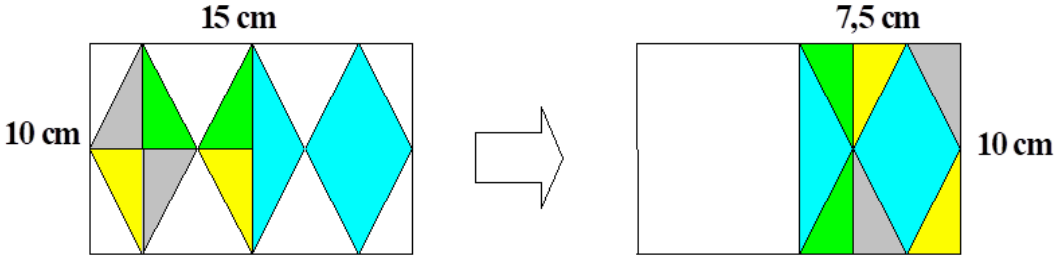
"Como mais se a pizza estiver dividida em 20 fatias"

V. divided two equal pizzas in 20 and 25 slices (Figure 4). He painted one slice in each pizza. Then he wrote "I eat more if the pizza is divided in 20 slices". His drawing, done freehand, was very accurate: precise circles, and their circular sectors, which seemed to have been drawn with a compass and a protractor to measure the angles. M. did not mobilize critical sense and elaborated an incomplete answer, which repeated a part of the mathematical data mentioned in the news: "There were less robberies because it changed from 1 in 25 homes to 1 in 20". She was not sensitive to the mathematical mistakes that were part of the news.

In Task B, where they needed to measure 1l of milk having just one 3l cup and another 5l one, V. used spatial reasoning and a graphic representation solving strategy. He drew the cups, representing the two consecutive moves when he poured the milk from the 3l cup into the 5l cup, mobilizing mathematical intuition. His drawings were very accurate, like the ones of the pizzas. M. used a mathematical composition as solving strategy and showed persistence in the task: she began with the 5l cup and had to do many moves to solve the task. They were both successful in Task C and mobilized abstract reasoning (AR).

In Task D, they were asked to calculate the painted part of the figure. The L1 clearly shaped their performances: V., whose L1 is Creole, preferred a global approach, geometrical reasoning, and a graphic representation solving strategy, that he briefly completed with a final arithmetic strategy (Figure 5). This decomposition is very rare, because it is much more complex than the most common one, that picks the three bottom triangles of the rhombus to the top part of the figure (César & Machado, 2024). M., whose L1 is Portuguese, preferred an analytical reasoning and an arithmetic solving strategy. She used the area of the rectangle (150cm2) and then divided it by 2, because the painted part (three rhombus) was half of that area.

Figure 5 – Task D – L1: Creole (V.)



$$7{,}5 \times 10 = 75cm^2$$

In Task E, V. preferred a global approach: he summed up all the merchant spent when he bought the painting, and then all he

earned when he sold it. Then, he subtracted one from the other, to calculate the profit. M. preferred a step-by-step approach: she imagined he had 1000 when he began (this was not mentioned in the text), which was a common resource when pupils mobilized concrete reasoning (CR). This need to use CR showed that M. was in the transition from CR to abstract reasoning. Then she followed the order mentioned in the text and subtracted or summed up according to what he did. The observation of the first week showed that M. had all the school materials, was organized and focused, while V. had no materials, but was organized and creative. Thus, they could both progress and learn from each other.

V. and M.'s dyad work: Can I still learn from you, even if I do not trust you?

The first content of the 9th grade was equations, a quite rejected content, particularly by those who experienced cumulative underachievement. Thus, the teacher/researcher decided to use a non-usual task, based on balanced scales. Some fathers of pupils participating in the Cape Verdean culture worked in food distribution to the market, and they knew these scales. So, they explained to their classmates that a balanced scale had the same weight on each side. Being the ones who knew these scales and who explained how they worked empowered them. So, this was another inter-empowerment mechanism used by their teacher/researcher. It was also meant to promote intercultural dialogue, celebrating diversity and different life experiences.

This task had six problems. Problems 1 and 2 were easier, because there was only one unknown weight. Problem 3 had two unknown weights, both on the same side of the scale. M. assumed leadership and talked to V. in a very pedagogical tone or criticized his comments when he needed "to see" before solving the problem. But he solved them so quickly and explained his reasoning and solving strategy so clearly that she was astonished. Her expectations were that he would not solve any mathematical task, would not even try, and would be poorly behaved. But the only one who had a less correct tone or words was she. Thus, V., who was described as "impulsive, with no control, or poorly behaved" (several school reports) was the one who showed more emotional maturity, avoiding conflicts, and not letting them escalate.

When the problems became more complex, because they had unknown weights on both sides of the scale (Problems 4, 5 and 6), V. and M.'s reasoning and solving strategies became quite different, and conflicts appeared, because M. was unable to accept V.'s solving strategies.

Figure 6 – Problem 4

1. M. – This one seems more complicated because there are weights on both sides.
2. V. – But the weights are all the same... so you can cut two on each side... and it gets easier... [He takes the sheet and cuts two weights on the right pan and two on the left pan]
3. M. – What nonsense is this? The sheet belongs to both of us, and I don't want to cut anything! [She is irritated] Sir, can you come here, please... [She changes her tone and puts her hand in the air, to get his attention. The teacher/researcher approaches them]
4. Teacher/researcher – So, what's the question?
5. M. – He cut weights on the sheet, and I don't agree. I didn't give him permission to do that [She says, in a less friendly and somewhat authoritarian tone]
6. Teacher/researcher – You need to agree, before writing on the sheet [M. smiles, victoriously], but you don't have to give him permission. You aren't in charge of him, nor is he in charge of you. If you have different ways of thinking and solving, write them both.

7. M. – But, to do that, we need to have two sheets, so I can do mine and he can do his...
8. V. – That would be if we did it individually, but we are working in pairs... [He says, softly, looking at the teacher/researcher, to see his reaction]
9. M. – But, if we don't agree, we can't work together. Each one must do his/her own!
10. Teacher/researcher – Each person does his/her own, in turns, on the sheet... on the same sheet [underlines 'the same', when speaking]. Then, each person explains his/her resolution to the other one and, thus, you both even know two ways of solving it. [He walks away, without giving M. time to argue]

(V. and M., O, audio recorded, September 1997)

As usual, M. assumed leadership and made the first comment (Talk 1). But V., who preferred a spatial reasoning, immediately thought he could use a simplification strategy, by cutting two unknown weights from each pan (Talk 2). M., who was described as "a very well-behaved pupil" (several teachers' reports), becomes angry and reacts in an aggressive tone, calling the teacher/researcher to complain (Talks 5, 7, and 9). V. had cut the weights with a pencil. It was easy to erase what he had done. But M. wanted to make her point, and to be recognized as the one in leadership, who knows Mathematics and the best solving strategy. So, she even uses an authoritarian tone when she states that she did not give him permission to cut some weights. If the teacher/researcher had reinforced her belief that V. needed her permission, this dyad would never have worked in a collaborative and dialogical way. But he reminds them of the rules of the didactic contract: they need to agree, before writing on the sheet, but none of them is in charge of the other (Talk 6). He also reminds them that if they have two different solving strategies, they write them both. M., who is not pleased at all to work with V., as she believes he knows nothing and will harm her top school marks, tries to break the didactic contract rules, claiming twice that they need separate sheets (Talks 7 and 9).

It is V., the one described in previous teachers' reports as "problematic and unbearable" that states they are working together, so they just need one sheet, engaging in the didactic contract (Talk 8). His lack of confidence makes him talk very low and look at the teacher/researcher to see his reaction. His Me-position (how the teacher/researcher saw him as a Mathematics learner) was not clear to him, and he needed reassurance. As explained by César (2009, 2013a, 2013b), the dialogical self of the students from vulnerable cultures, who experienced underachievement for many years, experiences several conflicts, for many years, even after achieving. Their life is quite challenging, in and out of school. Thus, for them, the way teachers act is much more important. In his last talk (Talk 10), the teacher/researcher repeats the rules, increasing V.'s positive self-esteem: they write in turns, on the sheet, and they explain to their peer how they solved the problem. As he did not want M. to go on arguing, he quickly leaves, not giving her the chance to argue again against those rules. His tone is calm, low, but firm, giving her no doubt that she needed to respect those rules.

After this episode, V. completed his solving strategy. He had only one unknown weight. Thus, he subtracted (1000 – 750) and wrote down that the weight was worth 250gr. Thus, he used a global approach, as he looked at the balanced scale as a whole, a spatial reasoning, and a simplification strategy, based on the manipulation of the variables, and an arithmetic solving strategy. M. preferred a step-by-step approach, analytical reasoning and an algebraic solving strategy.

$$1000 + 2x = 750 + 3x$$
$$1000 - 750 = 3x - 2x$$
$$250 = x$$

R: Each weight had 250gr

Then, they had to explain their reasoning and solving strategies. Each one of them could be asked to represent the dyad, in the whole-class discussion. For this, they needed to know how to do and explain both solving strategies. M. decides she would explain hers first. She is convinced that equations are "the real" Mathematics, and cutting weights is not (Talk 13). Thus, she assumes leadership and tries to decide what counts as Mathematics (Talks 11 and 13). She keeps undervaluing V.'s comments, even when they are interesting ones, which show he paid attention to the whole class-discussion (Talks 12 and 14). So, there are several negative evaluations in M.'s talks: "What nonsense is this?" (Talk 13) or "Maybe it could be […] But let me go on" (Talk 15). Even when V. said something clever, she did not trust his reasoning and knowledge. But V. wanted to try to learn Mathematics in this different class, in which he could "see" Mathematics. Thus, he keeps trying to use reasoning based on what he could see, and he realizes that the equal sign means the scale is balanced (Talk 12), and that the scale can be used as a metaphor (Talk 14). M.'s last question is a tricky one, because she knew he did not know how to solve equations. So, asking him this is a way of trying to stress that she is the one who knows Mathematics, the one that can solve equations.

11. M. – I translated the scale into an equation and, as the weights are all the same, I called each variable X. The

pan on the left corresponds to 1000 + 2x and the one on the right corresponds to 750 + 3x…

12. V. – So, the equal sign on the scale corresponds to it being balanced.
13. M. – What nonsense is this? An equation is Mathematics, it is algebra, it is not drawings, nor scales...
14. V. – I think the scale is a metaphor... like the one about the robberies [referring to the Task A, in the IACC], in the last class, the pizza was also a metaphor regarding the robberies, that's what the teacher said... Do you remember?
15. M. – Maybe it could be... We'll ask him later... But let me go on... Then, it was just a matter of solving the equation... Do you know how to solve equations?

(V. and M., O, audio recorded, September 1997)

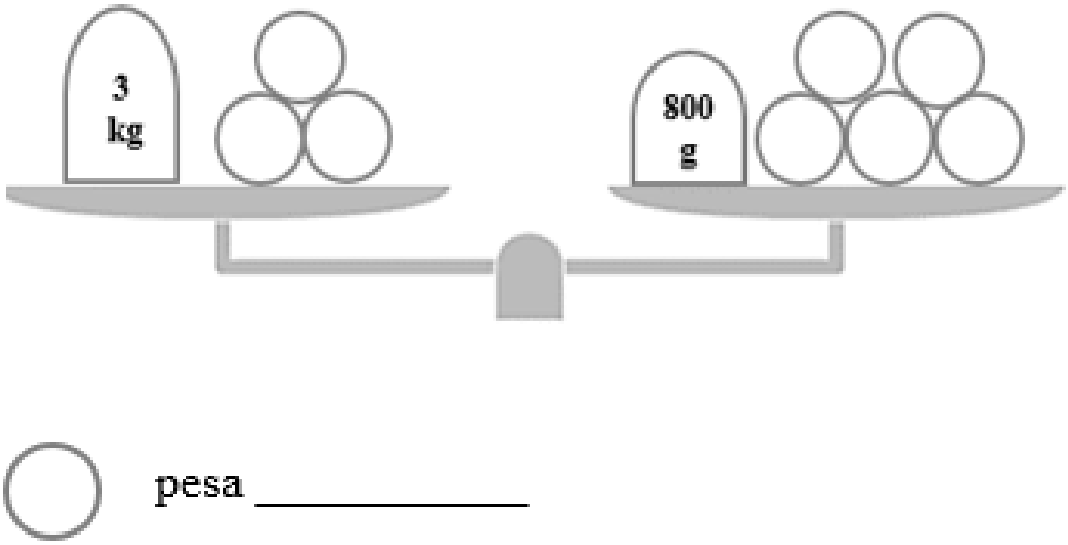
If V. did not feel empowered in the first week of classes, through the implicit messages and inter-empowerment mechanisms used by his teacher/researcher, he would immediately give up. But he reacts in a quite different way (Talk 16): he still states that he is a "zero at Mathematics", but this school year he has understood that if he could "see" Mathematics, he was able to do mathematical reasoning. He exemplifies that by telling M. what he saw in her resolution, and what rules to solve equations he deduced. This is an impressive example to illuminate the importance of respecting the preferred way to do Mathematics, particularly in the first weeks of classes. It also illustrates that PPS are essential to choose the first dyads and to empower pupils, as their teacher/researcher would never know V.'s potentialities, abilities and competencies if he had not evaluated them in the first week. Even M. is amazed with his ability to make connections between what he sees and the rules used to solve equations (Talk 17), as on short notice he is able to deduce some rules. But, as she does not trust or like him, she keeps praising him with tricky sentences, like "not completely dumb as I thought". Each praise comes with a negative comment, like when she says, "You must have done it well too" (Talk 19). She is not able to say, "You did it well too"; the doubt is still there, in the "must have done". It is a possibility, not a certainty, because M. is unable to believe that cutting weights is a valid solving strategy, accepted in a Mathematics class. But V. had a very striking characteristic: once he learnt something in a Mathematics class, he would no longer forget it. Thus, he went to the blackboard during the whole-class discussion, at the end of the class, and he was able to explain M.'s solving strategy, and the rules used to solve this equation. He explained the rules starting from what could be seen, and that helped a lot other pupils, whose L1 was Creole, to learn those rules. For them, "seeing" was at the origin of understanding, and only after that were they able to use more formal solving strategies.

16. V. – No. I am a zero in Mathematics. I had Level 1. But, this year, I realized that I know how to look and think through what I see and what I saw was that you changed the sign when you changed sides... I mean... it was + 2x and you put – 2x, when it changed side... Visually, that's what changed... But I didn't understand the 750, because that one had no sign and you also put – 750...
17. M. – You're not completely dumb, as I thought... That about the X is well observed... As for 750, if it didn't have a sign, it's because it's positive, it's a + sign. So, I did as you said: if you change to the other side – that's what it's called – it changes to the inverse operation. The inverse of – is + and the inverse of multiplication is division.
18. V. – And then, you did the Maths and X, which is the weight, gave you the same as me, because I also said that the weight was worth 250g. Therefore, if both resolutions are right, we get the same result...
19. M. – Well... I didn't think you could cut weights, but if it gave you the same result, you must have done it well too. We'll confirm when the correction is on the blackboard [she refers to the whole-class discussion]. I know how to explain yours. Can you explain mine?
20. V. – No big deal. I don't know how to solve equations, but I figured this one out.

(V. and M., O, audio recorded, September 1997)

Problem 5 was similar (Figure 7). So, V. suggested that they could try to use the other one's solving strategy (Talk 21). That means that he would use an algebraic solving strategy and M. a simplification strategy, complemented by an arithmetic strategy. This suggestion shows that V. was interested in learning, as he chose the solving strategy that was more difficult for him. M. accepted his challenge (Talk 22), but once again she does not clearly agree, she says she does not "even think it's bad". This illustrates that she is not convinced that working with V. will be beneficial for her. Up to now, if she could choose, she would change to another peer immediately. V. quickly starts writing the equation and explaining his reasoning (Talk 23), and M. says her first clear compliment (Talk 24).

Figure 7 – Problem 5



21. V. – Maybe, now we could do it the other way around: I would solve it with an equation and you would cut the weights, like I did. This way, we see if we really learnt the other one's resolution. What do you think?
22. M. – I hadn't thought about that, but I don't even think it's bad... Do you want to start?
23. V. – Yes. So... [Looks at the picture] Each weight is X... on the left side, we have 3kg, which is 3000 grams. Therefore, I write:

$$3000 + 3x = 800 + 5x$$

24. M. – Nice! You're able to translate the scale into an equation. Now, you just need to solve it... [V. goes on, quickly writing what is below]

$$3000 - 800 = 5x - 3x$$
$$2200 = 2x$$

25. V. – And now? Wait, let me think… [He goes back to the figure, takes a piece of scrap paper, cuts out the weights and sees that he must divide 2200 by 2] Ah! I've already understood… Here it is:

$$\frac{2200}{2} = x$$
$$1100 = x$$

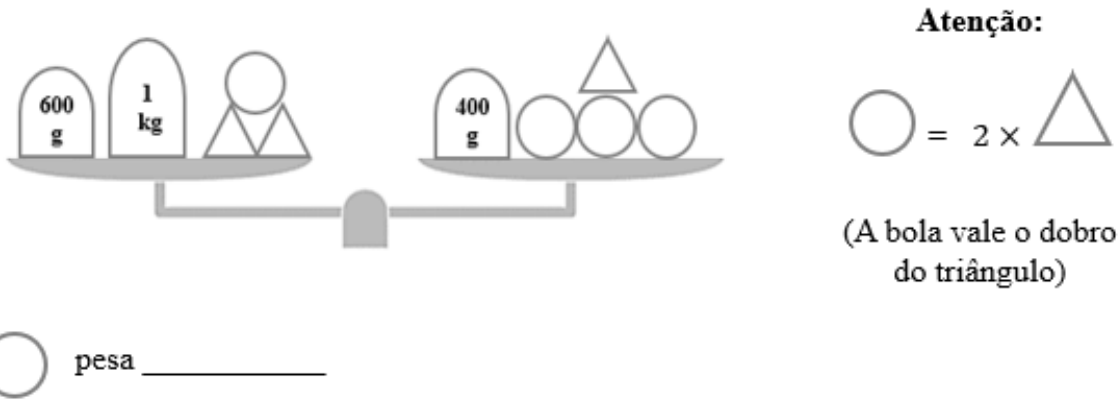
26. V. – As you explained to me in the other one [referring to Problem 4] that dividing is the inverse of multiplying, then 2x must mean... it is twice x, it is a multiplication... Is that it?
27. M. – Wow!... You're really smart! How did you have 1 in Mathematics?
28. V. – I thought it wasn't even worth trying...

(V. and M., O, audio recorded, September 1997)

The first steps are easy, and V. is quick. But then there is a new rule, so in Talk 25 he expresses his difficulty. But instead of asking M. how to solve that part of the equation, he asks her to give him more time to think. V.'s persistence in the task, and resilience, were never mentioned by his previous Mathematics teachers, but we were observing them in the first class of new Mathematics contents, and while working with a classmate who was not kind to him. Once again, this way of acting showed how important it is to design a different first week of classes, using inter-empowerment mechanisms. To deduce this new rule, he does what is usual: when faced with a stronger difficulty, pupils go back to their natural way of reasoning. Thus, he grabs another sheet, cuts the weights, and understands that he must divide 2200 by 2. Once he realized that, he was able to deduce the new rule (Talk 26), and even M. was amazed (Talk 27), and makes another compliment. She also asks him why he got Level 1, because she began realizing how quick he could learn (Talk 27). His explanation was that he "thought it wasn't even worth trying" (Talk 28), which illuminates, once again, how castrating it is to have natural ways of doing and learning Mathematics that are not accepted and valued by teachers (César & Machado, 2024, submitted, in preparation). It also illustrates the importance of PPS. Diagnostic tests, which are common in the first week of classes, only refer to knowledge, but they do not show pupils' mathematical abilities, competencies and potentialities. Consequently, they do not allow teachers to know how to begin constructing new knowledge and increasing pupils' development and learning.

The last problem has two types of variables, represented by circles and triangles (Figure 8).

Figure 8 – Problem 6



In Portuguese, it is written in Mathematics symbols and in words "Attention: The ball is worth twice as much as the triangle". V. states "I can only think about this one in my own way" (Talk 29). But he adds that he wants to learn M.'s way too. He suggests that each one will do it his/her way, and M. agrees, and asks him to start (Talk 30). He draws very quickly, solves the problem, and explains its resolution to M. in a few minutes (Talk 31). He is so fast and clear and shows such certainty that she is amazed once again. But she still thinks equations are what counts as Mathematics and tells him they are going to confirm his resolution using equations (Talk 32).

29. V. – I can only think about this one in my own way. Then I want to learn yours, but I don't know how to do this with equations. What if you did it with equations and I did it my way? What do you think?
30. M. – Okay. We do it on a draft sheet and explain. Then we move on to the answer sheet. Do you want to start?
    [V. draws the scales on the scrap paper but replaces each ball with two triangles. Thus, he only has one type of weight]
31. V. – I agree. So... each ball is worth two triangles, so I ended up with four triangles on the left and seven on the right, I can cut four on each side and I only have three triangles left on the right side. This makes it a lot easier... Then, I have 1600 grams on the left side, and, on the right, I have 400. Subtracting them leaves 1200 grams, which I will divide by the three weights and gives 400 grams each. Did you understand?
    (V. and M., O, audio recorded, September 1997)

M. decides she needs two variables and writes the equation (Talk 32). She solves it quickly up to the point in which she has two variables on one side and does not know how to go on. She asks V. if he knows how to solve this equation (Talk 33), but he does not. He only knows that she also had 1200, like him (Talk 34). This time M. is irritated, but with herself, not with V., and that was a first positive sign: she was no longer blaming him for their difficulties.

32. M. – I did. Now let's see if you did it right, confirming it with an equation. As we have two types of weights, we must have two variables: x and y. Triangles are x and balls are y. Here it is:

$$600 + 1000 + 2x + y = 400 + x + 3y$$
$$1600 - 400 = x - 2x + 3y - y$$
$$1200 = - x + 2y$$

    [M. stands still, thinking. There is a long moment of silence]
33. M. – And now? I don't know how to solve this... Do you?
34. V. – Me? Do you think so? [Smiles] But there's one thing I know: it also gave you 1200, like me, before dividing by the three triangles that I had left on the right side...
35. M. – Yes, that's true. But what do I do with those two variables? What bullshit! [Irritated, but with herself. They remain silent for a long while. She is looking at the equation and he at the figure]
    (V. and M., O, audio recorded, September 1997)

As M. could not come up with a solution, V. decided to suggest something (Talk 36). He does it in a very humble way, saying that maybe this is "too silly", and reminding her that she was the one who knew equations (Talk 38). He always tried to avoid making her feel uncomfortable, because he wanted to avoid conflicts, and he knew she was not pleased to work with him, as he stated in informal conversations and in the first interview. But he also thought that "no top pupil would choose me as his/her peer" (V., I1, December 1997). So, he kept a low profile and did not try to lead, something he usually did in other contexts (e.g., school breaks). As M. does not have any other idea, she agrees they should try (Talk 39), and writes the equation, using just one variable. M. could quickly solve the equation, and the result is the same V. got. He states that (Talk

40), but even so, M. is not sure about what she did (Talk 41). She was only convinced when their teacher/researcher praised them, in the whole-class discussion, because they had two successful resolutions. Then, she said "We were the best dyad. It was better than I thought to work with you" (M., IC, teacher/researcher's diary, September 1997).

36. V. – Maybe this is too far-fetched... too silly... But we could just use x. If the balls are worth two triangles, on the left we have 4x... each triangle is an x and we have two... and each ball is worth 2x... and on the right we have 7x... and it's already similar to my drawing... [He looks at her]
37. M. – I don't know... Do you think so?
38. V. – You're the one who knows equations. Do you have any better idea?
39. M. – No. So, let's try...

$$600 + 1000 + 2x + 2x = 400 + x + 6x$$
$$1600 + 4x = 400 + 7x$$
$$1600 - 400 = 7x - 4x$$
$$1200 = 3x$$
$$\frac{1200}{3} = x$$
$$400 = x$$

40. V. – It gives the same result as in my way…
41. M. – Well… maybe it's fine… but I don't know if it is…

(V. and M., O, audio recorded, September 1997)

Despite their success solving these problems, there were still spatial signs of their distrust in each other: they accepted to be part of the same dyad, and to sit at the same table, but their chairs were as far as they could be to share the same table. These spatial positions only changed in the next class, after an episode discussed by César (2009, 2024) and César and Machado (2024). But if we compare these episodes with the one analysed by César (2013b), when the pupils could choose their peer in the last class of that school year, we realize how much V. and M. were fond of each other. M. also states the changes she experienced in her dialogical self, after working with V. and with other classmates. She states there was a before and an after this school year. She recognizes cognitive progress, as well as emotional and social benefits.

> Before, I only thought about myself. I didn´t care if the others understood what I had done... and I didn't think I could learn anything from my classmates. I only thought I should listen to the teacher, do the exercises, and study a lot at home, to memorize everything. (...) Now it's different and the truth is that, in the beginning, I was furious when I knew I would sit next to V., because he was very poor in Maths. But right now, I just wish I could sit next to him in other classes... He is so smart and has such a fantastic way of analysing the questions... he is so critical about everything he reads and listens and does (...) Thanks to him I even developed some abilities I did not have... or I did not know I had... Well... maybe the truth is that I only thought I had to repeat everything, and now I must also think about what I'm doing... to understand what I'm asked... and to exchange ideas with my colleagues, teaching them, and learning from them... Now I'm not only competing, trying to be the best pupil in class... I'm also helping, being helped, discussing... (...) Finally I could have some real friends in my class (M., I1, December 1997)

Before, she describes an obedient and engaged pupil, but with no critical sense, believing that learning is just memorizing and repeating. This shows she had very traditional beliefs about a pupil's role. She had no conscience about the others, she was convinced she could not learn from any other classmate, she was very competitive, and somehow lonely, because she confesses that now she has some real friends in her class. She also states that now, if she could choose, she would sit next to V. not only in Mathematics, but also in other subjects, and she mentions some of his characteristics that she values. In high school (10^{th} to 12^{th} grades), although in a different class, as V. went to Arts, and she was in the Scientific branch, sometimes they studied together, because the first term was about Geometry.

M.'s father stated in an informal conversation that "I was very suspicious about dyad work, having M. sitting next to poor pupils, working with them. But now I realize how much she has progressed, she is better prepared for high school than if she had had traditional classes" (M.'s Father, IC, June 1998). This is strong evidence because he was convinced that collaborative and dialogical work was only adequate to those who underachieved. In the first meeting with her parents, he even suggested that the four top-mark girls should only work with each other. The next school year he adds, "A whole term learning Geometry was hard. But she was lucky: V. studied with her. He came to our home. He is so smart, creative, has a sense of humour, and is so polite. They became good friends" (M.'s Father, IC, April 1999). This is an illustration of the impacts of collaborative and dialogical work on socialization, as it does not only shape pupils' life trajectory of participation, but also has impacts on their parents' expectations about different people, like those participating in other cultures. Thus, participating in the IK had cognitive, social, and emotional impacts. It facilitated intercultural dialogue. Together with bi-univocal cultural mediation (César, 2017; César & Machado, 2024, submitted), developed in that school, it contributed to an

intercultural and inclusive education.

Many pupils participating in this class also stated the IK impacts when they answered the questionnaires, in January (Q2), in June (Q3), and in the follow-up (Q4 to Q13). The first impact they described was that "this kind of work develops pupils' abilities" (V., Q2, January 1998). Later, other issues were mentioned, related to socialization and citizenship: "to learn how to respect other people's opinion" (M., Q5, June, 2000), "to know how to live together and respect each other" (D., Q7, June 2002), "to learn other points of view and how to discuss our own" (G., Q10, July 2005) and "I think this way people work with more freedom, explaining their opinions and their doubts, respecting everyone's ways of doing Mathematics and living their life. Finally, we had a place in school, we felt we belonged there too" (W., Q12, June 2007). Quite often, when they were already attending college or professional courses (last two excerpts), they stressed how useful what they learnt in the IK was for their life trajectory of participation, and showed they were able to use those abilities and competencies in an autonomous way, in other contexts, scenarios, and situations.

It is important to be aware that the way they lived school achievement was quite different for V. and M., as mentioned by César (2009, 2013a, 2013b, 2024). M. wanted to go to Medicine. Achieving top marks was very important for her, particularly in high school and the national exams of the 12th grade, because there are *numerus clausus* in Portuguese universities. She felt no conflicts in her I-positions (e.g., daughter, pupil, friend) and in her Me-positions (how the others see me, and how I think they see me). Both her parents had degrees, they wanted her to go to college, other members of her family also had degrees or were already in college, and her friends had similar expectations. Thus, for her, achieving was praised by everyone, in the different contexts where she lived.

For V., achieving came with many conflicts, because his family wanted him to start working after the 9th grade, and that was also what he expected to do, as he wrote in Q1 (September 1997). They were poor, and needed the money he could earn. His mother stated that "a diploma does not put food on the table" (V., I1, December 1997). They thought their life would always be difficult, and School was not for them, "it was meant for rich people, who could afford long-studies" (V., I2, June 1998). Thus, when V. tried to discuss alternative school paths, his family did not support him, and his friends and neighbours also tried to convince him to start working. But, after experiencing so much success in Mathematics, feeling accepted at school, and having so many friends among his classmates, he decided to go to high school. He had doubts: marks changed to a scale from 0 to 20; teachers and the curricula were more demanding; his 9th grade class would be divided, as they could choose different branches; and he thought he had those marks because he had that "weird" teacher/researcher, who believed in him. But M., and other classmates, thought he was "too clever to stop studying" (M., I2, June 1998).

After many internal and external conflicts, he decided it was worth a try. As he put it, "to begin working in a poorly paid job I'm always on time" (V., I2, June 1998). But it was only many years later, when he had a stable job he loved, as an architect, a family of his own, and better relations with his parents and siblings, that he felt he had made the right choices. César (2009, 2013a, 2013b) stated that if the IK had not included the follow-up, and if pupils had not gone on sending news even when the formal follow-up had ended, she would have doubts about the advantages of the IK for pupils participating in vulnerable cultures. Their life trajectories of participation included so many barriers and struggles, that having access to the follow-up data was essential to understand the impacts of the IK. Like for so many other pupils participating in vulnerable cultures, V.'s life trajectory of participation was much more difficult (César, 2009, 2013a). For many years, he felt that "being poor, black, from a slum area, made me have to prove much more than those who are rich, white, and live at fancy homes" (V., I9, July 2005).

When he finished the 12th grade, he decided to rent a room in Lisbon. He had better conditions there: piped water, electricity, a washing machine, a fridge, a toilet, a room just for himself, he was close to his job, and later to college. Being able to walk between them saved money and time, two important issues. He described that room as a place to rest and feel at peace: "I can eat, sleep, read, rest, when I want. To me, after where I lived, it looks like a palace" (V., I5, June 2001). But he also felt lonely many times, he missed his family, the parties, the food, a neighbourhood where he knew everyone. Thus, he had a job, a room, was admitted to college, but he also had to struggle for every small step. He discovered subtle forms of racism: the difficulties to rent a room or a flat, even when he had a contract to prove he could pay them; the university teachers that did not believe such good work was his own, because he was not from a family of architects, and his family name carried no weight; or clients' acceptance of his projects when he became a graduated architect. As he told us, "I realized that racism has many faces, and many forms, some of them so subtle that we need time to realize what is going on" (V., I13, 2009). But in the same interview he also stressed that "My life is even better than I ever dreamed of when I was in high school" (V., I13, July 2009). V. started working in an *atelier* when he finished high school. He began by doing easy tasks: going to the post, buying what was needed, and helping with other stuff. His bosses encouraged him to read a lot, they lent him expensive books, discussed with him new tendencies in Architecture, told him to see exhibitions. It was a new world that entered his life. Then, when he finished graduation (July 2005), his bosses already knew his abilities and competencies quite well, and he was

promoted. He decided to do a 6-year course in four years. He wanted to have a degree as soon as possible, to become an architect, so that he could only work. He had been working and studying at the same time since the 10th grade, and he was tired of that life. So, by 2009 he had achieved a stable life, his own home he had designed, a family, and children. It was by then that he told us he knew he had done the right choice when he went on studying.

V. and M. went on being friends. This was something common: they kept friends from secondary school, and were in touch with many others, through social media. When the economic crisis was stronger, many of them helped colleagues to go abroad, or find another job, as they told us in the interviews, or by email. Thus, the impacts of the IK project went far beyond what we had planned. Something similar happened during the pandemic, an unexpected issue that affected us all, and in which innovative solutions were needed more than ever. At the time, having many colleagues with whom they were still in contact helped them face those challenges. V. emigrated in 2014, due to the economic crisis. During the pandemic he helped some colleagues to emigrate too, feeling useful to those who had lost their jobs, and even their homes. Many of them wrote us during the pandemic, sending news, asking for opinions, discussing ideas to help those in need. By then the formal follow-up had ended, but autonomy was a major value for the IK team, and pupils proved they had internalized it: nowadays they contact us on their own, when they think they should do so.

FINAL REMARKS

Before starting the IK, its coordinator was already aware that knowing pupils' abilities and competencies was essential to promote school achievement. But, despite being a psychologist, she did not want to use psychometric tests or scales. Thus, the IK team elaborated a developmental instrument, the IACC, that allowed them to picture pupils' potentialities, abilities, and competencies related to Mathematics. This instrument, conceived by psychologists and Mathematics teachers, was meant to be used and interpreted by teachers, increasing their agency, autonomy, and responsibility. This was a solid difference regarding the instruments that existed until then, which were meant to be used and analysed by psychologists.

The way teachers operationalize the curricula is an important feature to increase, or block, school achievement, particularly for pupils who participate in vulnerable cultures. The IK team realized how important it was to design an out-of-the-ordinary first week of classes when the aim is to negotiate a different didactic contract (César, 2009, 2013a; César et al., 2014). Teachers used non-usual tasks (e.g., TIPT1, IACC) to promote pupils' engagement, their positive self-esteem, and to get to know pupils' potentialities, abilities, and competencies. They used empowering implicit messages, and inter-empowerment mechanisms. The most impressive experience was the whole-class discussion of the IACC, in which each pupil went to the blackboard to explain a successful solving strategy, registered by the other classmates in their notebooks. Watching every pupil succeed while doing Mathematics, even those who had experienced underachievement, was a transformative experience. It contributed to empower pupils, encouraged them to express their voices, and favoured social justice.

The IK team wanted to disseminate practices based on collaborative and dialogical work, as they proved to be effective in the promotion of an intercultural and inclusive education, increasing equity and social justice. Before participating in the IK team, teachers were engaged in pre- or in-service teacher education workshops, based on simulations, case study analysis (including videos), and role-playing. Thus, they prepared themselves to use and interpret the TIPT1, Q1, IACC, and observation. The IK team was aware that teacher education, particularly preparing teachers to act as decision makers, and developing their epistemological conscience (César, 2010), was a major issue. The IK team used research to promote learning, development, and empowerment, even among researchers, teachers and pupils.

The analysis of each action-research project soon began and was disseminated in scientific events and papers. But it was after the end of the follow-up (July 2016) that the meta-analysis of the IK's complete *empirical corpus* began, and some important issues were discussed: (1) the construct of bi-univocal cultural mediation (César, 2017; César & Machado, 2024, submitted); (2) how the L1 shapes pupils' approaches, reasoning, and solving strategies (César, & Machado, 2024, submitted, in preparation); and (3) the concept of Personal Performance Shapers (PPS), discussed for the first time in this paper.

When they are characterizing a class, teachers focus on pupils' school marks. Thus, they only consider two groups: the achievers; and the underachievers. But the abilities and competencies that pupils mobilize and use in an autonomous way are also an important dimension. In the concept of Personal Performance Shapers (PPS), whose visual representation is a two-axes reference frame, there are four different groups, represented by each quadrant. This concept allows teachers to have a more accurate picture of each class, represented by its point cloud (e.g., Figure 2). One can realize, from among those who are underachieving, which ones mobilize abilities and competencies that facilitate their quicker access to school achievement (Quadrant 4), but it also shows, from among those who are achieving, which ones need to develop more abilities and competencies (Quadrant 2). The fewer pupils a class has in Quadrant 3, the easier it is to form the first dyads. This concept

is based on a developmental perspective. Thus, it illuminates potentialities, favouring pupils' development and learning. The PPS also allow teachers to identify the tasks that are the best to each class to facilitate pupils' work in their ZDP (Vygotsky, 1932/1978), and to act as more competent peer in alternating moments. Thus, the PPS facilitates teachers' conscious choices regarding practices. Using a different first week favours underachieving pupils' legitimate participation, and their school achievement. Transitions from a dichotomic perspective (achievers; underachievers) into a more complete concept, like PPS, is a major issue to promote pupils' engagement, positive self-esteem, agency, and achievement. So, it should be considered in pre- and in-teacher education.

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