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Respecting Cultural Diversity in Mathematics Classes

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Abstract **Original Research Article**

We live in intercultural societies and schools. In schools there are pupils with distinct mother languages (L1) that shape their mathematical performances (César, 2024b; César & Machado, 2024, submitted). Pupils whose L1 is ideographic, like Creole (Cape Verde), usually prefer global approaches to problem solving, spatial or geometrical reasoning, and graphic representations as solving strategies. Those whose L1 is phonetic, like Portuguese, usually prefer step-by-step approaches to problem solving, analytical reasoning, and arithmetic or algebraic solving strategies, the ones most teachers value more. The rejection of their natural approaches, reasoning, and solving strategies makes pupils who participate in Cape Verde culture lose their voice, underachieving in their mathematical performances (César, 2009, 2013a, 2024a, 2024b; César & Machado, 2024, submitted). The *Interaction and Knowledge* (IK) project lasted 12 years and involved around 600 classes, taught by 69 Mathematics teacher/researchers who participated in the IK team. The IK used collaborative and dialogical work, implicit messages, inter- and intra-empowerment mechanisms (César, 2009, 2024b), and bi-univocal cultural mediation (César, 2017) as tools to give pupils a voice and to increase their mathematical performances, contributing to an intercultural and inclusive education. We focus on data from the action-research projects (Hamido & César, 2009) and we assume an interpretative paradigm (Denzin, 2002). Data treatment and analysis was based on a narrative analysis (Clandinin & Conelly, 1998). The analysis of some examples of answers given to an instrument to evaluate pupils' abilities and competencies (IACC), elaborated by the IK team, as well as the performances regarding some tasks used in class, illuminate how important it is that Mathematics teachers respect pupils' approaches, reasoning, and solving strategies. This is an essential feature to empower them, to promote equity and social justice, thus contributing to an intercultural and inclusive education.

Keywords: Mother language (L1), Mathematics learning, inter- and intra-empowerment mechanisms, life trajectories of participation, bi-univocal cultural mediation

INTRODUCTION

A society and school that are increasingly more intercultural bring new challenges, particularly as regards power and the way it is, or is not, distributed among participants. Although the Constitution of the Portuguese Republic (AR, 2005) considers all citizens equal before the law, a study that lasted 14 years (1996-2000), carried out in Portugal, identified the existence of racial discrimination on the part of police forces (Alves, 2023). It showed that 36% of deaths due to police intervention affected non-Caucasian people. A Roma person was 43 times more likely to be killed by the police and a black person 21 times more likely. These data are more impressive when we realize that, according to the 2020 Census (INE, 2021), only 0.4% of the population living in Portugal is of gypsy ethnicity and 0.9% is black. Therefore, negative discrimination exists and people who participate in these cultures deal with it daily, as also reported by César (2009, 2013a, 2013b, 2017, 2024b) and César and Machado (2024, submitted).

In the latest education statistics available, referring to 2021/22 (DGEEC, 2023), underachievement rates among pupils coming from abroad have increased and are 30% higher than those of non-immigrants. A study by the General Directorate of Higher Education (MCTES, 2024) reveals that more than half of poor pupils do not attend higher education, with only 44.35% getting there. Of these, 6% of pupils who benefit from Social Security Scale A (the highest) attend excellence courses, requiring the highest admission averages. These data illustrate the lack of equity and social justice in education.

One of the most represented cultures in Portuguese schools is the Cape Verdean. Often, they are already second or third generation, so they are Portuguese. However, at home and in the neighbourhood where they live, with friends and family, they speak Creole. It is only at school and in specific situations (e.g., shops, at work) that they speak Portuguese. It is in Creole that they think, count, sing, express feelings, and dream. Sometimes, when they start school, they do not master the Portuguese language, which creates additional difficulties for them (César, 2017; César & Machado, submitted). Ideographic mother tongues (L1), such as Cape Verde Creole, or phonetic languages, such as Portuguese, shape different approaches to problems, mathematical reasoning, and solving strategies (César, 2024b; César & Machado, 2024, submitted). These ideographic languages do not have singular and plural, nor masculine and feminine for nouns, adjectives, articles, and pronouns. Verbs do not have conjugations with different endings, depending on the subject, and words take on different meanings depending on their place in the sentence. For many years, Creole only had an oral tradition. The language was learnt from the elderly, without resorting to reading and writing. Phonetic languages, such as Portuguese, have had a written basis for many years. They have singular and plural, masculine and feminine, verbal conjugations with endings that vary with the subject and verb tense. When the L1 is Creole, pupils tend to prefer a global approach to problems, spatial and geometric reasoning, and graphic representation solving strategies (e.g., diagrams, drawings, graphs). Battista (2007) differentiates spatial reasoning and geometric reasoning. All geometric reasonings are also spatial ones. But there are spatial reasonings that are not geometric. If pupils' L1 is Portuguese, they prefer a step-by-step approach, analytical reasoning, and arithmetic, algebraic or mathematical compositions as solving strategies (César, 2024b; César & Machado, 2024, submitted).

Several authors studied the importance of language in Mathematics learning (Civil & Hunter, 2015; Nieto, 2018; Planas, Morgan, & Schütte, 2021), but none of them addressed the issue of how L1 shapes mathematical performances and learning. As César (2024b) and César and Machado (2024, submitted) underlined, if pupils' natural approaches, reasoning and solving strategies were accepted in Mathematics classes, none of these cultural groups would be at a disadvantage. However, that is not what happens. As most Mathematics teachers are Portuguese and have a phonetic L1, they favour the approaches to problems, reasoning, and solving strategies they feel most comfortable using. Often, the natural ways of solving problems for pupils whose L1 is Creole are discouraged or even forbidden. As claimed by Abreu, Bishop, and Pompeu (1997), Apple (1999), César (2024b), César and Machado (2024, submitted), and Reinke, Stephan, and Cobb (2024), one of the ways of exercising power is to decide what is considered legitimate knowledge and what is not accepted as such. This happens both in classes and in other contexts, scenarios, and situations where power issues are usually little discussed, or even silenced: scientific journals, in relation to the reviews that are made (Apple, 1999); or scientific events and research (César, 2010).

Power, voice, and participation

In every situation where there are social interactions and decisions, there are also power issues that are at play and shape performances (Beccuti & Robutti, 2022; César, 2009, 2013a, 2013b, 2017, 2024b; César & Machado, submitted; Civil & Hunter, 2015; Ernest, 2024; Skovsmose, 2023). The way in which this power is used, from top to bottom, or distributing it among the different participants, defines the type of community in which we work, as well as the form of participation: (1) legitimate participant; or (2) peripheral participant (César, 2009; Lave & Wenger, 1991). Legitimate participants feel they have a voice that is listened to, that they can express themselves, and that their culture and person are respected and appreciated. They tend to develop a positive (academic) self-esteem, positive social representations (e.g., of themselves, as pupils), resilience and to trace life trajectories of participation that include long studies. Peripheral participants do not feel appreciated or respected. At school, they often stop trying to solve tasks and, in more extreme cases, they end up dropping out early.

Both in vulnerable minorities who need specialized educational and social support (César, 2014; César, Machado, & Ventura, 2014; César & Machado, submitted), and in those who participate in socially undervalued cultures (César, 2009, 2013a, 2013b, 2017, 2024a, 2024b; César & Machado, 2024, submitted), for peripheral participants to have a voice, becoming legitimate participants, it is essential to use inter-empowerment mechanisms from the first classes. Later, they can be internalized, becoming intra-empowerment mechanisms, used autonomously in other contexts, scenarios, and situations (César, 2009, 2013a, 2014). Inter-empowerment mechanisms should be used in families since babies are born, as exemplified by César (2009, 2013b) and César and Machado (submitted). If, before entering school, children have already internalized intra-empowerment mechanisms they mobilize autonomously, their performances and social interactions with classmates and teachers are facilitated (César & Machado, submitted). When families do not use inter-empowerment mechanisms, it is even more important to use them in schools, in and outside classes, to promote pupils' mathematical performances and positive self-esteem (César, 2009, 2013a, 2013b, 2017, 2024a, 2024b). Using these mechanisms allows pupils to develop positive social representations of Mathematics and of themselves, as learners (Machado & César,

2013). These inter-empowerment mechanisms play an important role in the dialogical self (César, 2013b, 2017, 2024b; César & Machado, submitted).

Hermans (2001, 2003) conceived the dialogical self with many voices (Bakhtin, 1929/1981), which express each I-position (e.g., pupil, son/daughter, friend, classmate, professional, mother/father, husband/wife) and Me-position (how the others see me; how I think I am seen by them). These different I- and Me-positions are sometimes in conflict which means they are dialogical. These identity positions are organized into a diversity of architectures at different moments, with differentiated dominant I-positions (e.g., when answering to an assessment test in class, the pupil I-position is dominant). Sometimes, there are conflicts between the different I-positions (e.g., when solving a task on the test, the pupil I-position should be dominant, but if s/he is thinking about her/his girl/boyfriend, that I-position becomes dominant and in conflict with the pupil I-position). As illustrated by César (2009, 2013a, 2013b, 2017, 2024b) and César and Machado (submitted), conflicts between I-positions may also be due to participation in different cultures (e.g., Cape Verdean culture and school culture), with different values, ways of acting, and rules that are not always compatible. These conflicts can be acute, causing unbearable suffering, or creating doubts about decisions regarding the future (César, 2009, 2013a, 2017; César & Machado, submitted).

In the previous paragraph, we used the word culture in two different senses: (1) culture in a large sense (Cape Verdean culture); and (2) culture in a strict sense (school culture). In the first sense, the L1 is one of the fundamental features of the culture and we use a capital letter to designate the culture and the language. L1 is shared by those who participate in that culture in a large sense and clearly contributes to their sense of belonging. Culture in a large sense is made up of traditions, values, social rules (e.g., how to eat or dress), rituals, and religion. Anyone who participates in this culture can agree with, and practice, all of them, or just part of them. However, if you stray too far from what characterizes that culture, you will no longer be recognized as participating in that culture. In a strict sense (e.g., school culture), culture is not characterized by the language of instruction. Many schools have the same language of instruction and different school cultures. As mentioned by César (2014) and César and Machado (submitted), what characterizes cultures in a strict sense are the ways of acting and seeing the world, understanding the rules, sharing meanings and experiences, among the participants.

The more a school culture distributes power among the different participants, allowing each one of them to have a voice and act as a legitimate participant, the more intercultural and inclusive it becomes (César, 2009, 2013a, 2013b, 2017, 2024b; César & Machado, submitted). Generally, this movement of distributing power begins with those who hold it, such as the board of directors and teachers, at school. If a teacher wants to use power in an authoritarian way, it is more difficult for pupils to express their voices. Many may adopt a submissive position, which blocks development and learning. However, some pupils' actions, led by those who mobilize more intra-empowerment mechanisms, manage to make their voices heard and change the school culture. César and Machado (submitted) report an example of a blind pupil, in a school with several blind pupils, but few teachers who knew Braille, who managed to get the teachers to attend a workshop, becoming more sensitive to the needs of the blind pupils. But this seldom happens. Differences in age and instruments for acting upon each other (e.g., assessment of pupils by teachers and not the other way around) do not make this a common process.

Collaborative and dialogical work, equity, and social justice

Collaborative and dialogical work, particularly in dyads and small groups, followed by a whole-class discussion (César, 2009, 2024b; César & Machado, 2024; submitted; Staples, 2007), promotes pupils' mathematical performances. Social interactions shape mathematical performances (César & Kumpulainen, 2009; Cobb, 2012; Cobb & Hodge, 2007; Giglio & Arcidiacono, 2024; Ligorio & César, 2013). As Sfard (2008) claimed, thinking is communicating. We conceive collaborative work as dialogical, where cognitive conflicts, confrontations of reasoning and solving strategies, or debates on antagonistic positions, contribute to learning (César, 2009, 2013a, 2013b, 2024b; César & Machado, 2024). Collaborative and dialogical work allows pupils to have a voice, expressing their agency (Giglio & Arcidiacono, 2024), improving their academic self-esteem, developing abilities, competencies, and new social representations, which are also dialogical, according to Marková (2005). Working collaboratively in their Zone of Proximal Development (ZPD), which is between actual development and potential development (Vygotsky, 1934/1962), pupils transform potential abilities and competencies into real development, that they mobilize autonomously. To facilitate this process, teachers need to know the pupils' abilities and competencies from the first week of classes, to choose, adapt or elaborate tasks in which pupils work in the ZPD and in turns assume the role of the most competent peer (César, 2009, 2013a, 2024b; César et al., 2014; César & Machado, 2024). Thus, collaborative and dialogical work is even more powerful than Vygotsky (1934/1962) mentioned, as both pupils progress. Subsequently, it is beneficial for them both.

César (2009, 2013a, 2013b, 2014, 2024b), César and her collaborators (2014), and Rose (2002) highlight that the curriculum can be used as a vehicle for inclusion, or for exclusion, depending on how it is operationalized by teachers. If the didactic contract, governed by more implicit than explicit

rules (César, 2009; Schubauer-Leoni, 1986), is negotiated with pupils, distributing power (Apple, 1995; César, 2013b, 2024b), it allows them to have a voice and act as legitimate participants (César, 2009, 2013a, 2024a, 2024b; Lave & Wenger, 1991). But the tasks must also be adequate to the pupils' abilities and competencies, their interests, and needs. The evaluation process must be coherent, rich, and differentiated, prioritising moments of individual, dyad, and group evaluation, oral and written, with diverse evaluation instruments (e.g., mini-tests, tests, mathematical compositions, project work, problems, and exercises). If power is imposed from the top down, silencing pupils (peripheral participants), not making it easier for them to mobilize and develop abilities and competencies, it is contributing to exclusion. The same happens if the way the curriculum is operationalized favours the dominant culture, to the detriment of other vulnerable minority cultures, that is, if it does not favour equity and social justice in Mathematics education (Apple, 1995; César, 2009, 2013a, 2013b, 2024b; César & Machado, 2024, submitted; Civil & Hunter, 2015; Nieto, 2018).

To promote an intercultural and inclusive education, which respects equity, it is essential to create thinking spaces. Perret-Clermont (2004) defines them as spaces in which participants feel safe to raise doubts, questions, or discuss diverse solving strategies. Thinking spaces are part of something essential in collaborative and dialogical work: creating learning communities (Civil, 2007), which favour respect and appreciation for diversity, promoting equity and social justice. However, to develop these learning communities, we need to change practices during the first week of classes, facilitating a break with previous practices ruled by a traditional didactic contract, where vertical social interactions (teacher/pupil) prevail at the expense of horizontal ones (pupil/pupil), assuming only pupils learn from teachers, or only pupils with learning difficulties from those who achieve high academic success. When we design a first week dedicated to getting to know pupils, using it to give them a voice and to convey such powerful implicit messages as "everyone is capable of learning Mathematics" and "you can learn Mathematics with your colleagues", we are facilitating the development of thinking spaces (Perret-Clermont, 2004) and learning communities (Civil, 2007).

METHOD

The *Interaction and Knowledge* (IK) project lasted 12 years (1994/95-2005/06) and included a 10-year follow-up (César, 2009; César & Machado, submitted). This follow-up ended in July 2016 and was followed by the meta-analysis of the full *empirical corpus*, which is very rich and thus still ongoing. From this meta-analysis emerged new theoretical knowledge: (1) the construct of bi-univocal cultural mediation, coined by César (2017); (2) how L1 shapes pupils' approaches to problems, mathematical reasoning, and solving strategies (César & Machado, 2024, submitted); and (3) the concept of Personal Performance Shapers (PPS), coined by César (2024b).

Three research designs were used in the IK project: (1) *quasi-experimental* studies; (2) action research projects; and (3) case studies (Hamido & César, 2009). We assume an interpretative paradigm (Denzin, 2002) and focus on data from the action-research projects and Mathematics classes. We consider two research questions: (1) How does L1 shape pupils' mathematical performances? and (2) What are the impacts of collaborative and dialogical work in pupils' mathematical performances?

The participants are the pupils from around 600 Mathematics classes (all over Portugal and two Cape Verde islands), their families, 69 teacher/researchers, four psychologists (researchers), other educational agents, and internal and external observers. The diversity of the participants (who are informers speaking in the first person) allowed for the triangulation of the sources and the researchers' interpretations. Pupils also collaborated in the triangulation of the interpretations when their excerpts were used, as giving voice to the participants is a primary feature of intercultural and inclusive research (César & Machado, 2024, submitted). To keep pupils anonymous, they chose their own fake names (e.g., Tatiana and Tomás).

The data collecting instruments used in the first week of classes are: (1) a task inspired in projective techniques (TIPT1); (2) a questionnaire (Q1); and (3) a developmental instrument to evaluate pupils' abilities and competencies (IACC), composed of five tasks (César & Machado, 2024, submitted; Machado, 2014). Later we used other questionnaires (Q) and tasks inspired in projective techniques (TIPT), informal conversations (IC), in person and via e-mail, observation (O), audio and video taped, teacher/researchers' diaries (D), internal and external observer reports (IR and ER), and documents (DOC). The variety of the data collecting instruments allowed for their triangulation.

In each class we collected data during a whole school year. When the teacher/researcher was already from the school board, that class participated in the IK project for a cycle (e.g., 7th to 9th grade), like Tatiana and Tomás's class, that we discuss in this paper. In the follow-up we collected data for 10 years and we began it when that class was no longer taught by a teacher/researcher from the IK team. At the end of the school year (June/July), the pupils answered a questionnaire and six of them, in each class, were interviewed. Before starting data collection, the first authorization was provided by participants,

or by their guardians and themselves when they were minors. But in the IK team we assume a more demanding notion of informed consent. Thus, when we used any excerpt that person gave authorization to use it, as well as regarding the interpretation we wrote in that paper. They discussed that interpretation with us and made suggestions. Thus, research and writing were also collaborative and dialogical. To the IK team this way of acting is an important feature of intercultural and inclusive research (César, 2013b, 2024b; César & Machado, submitted). Data treatment and analysis was based on a narrative content analysis (Clandinin & Connelly, 1998), from which inductive categories emerged. This analysis allowed us to trace pupils' life trajectories of participation.

RESULTS

In the first week of classes teacher/researchers did not explore new mathematical contents. This week was used to get to know pupils better, using TIPT1, Q1, and a developmental instrument to evaluate pupils' abilities and competencies related to Mathematics (IACC). The analysis of these three data collecting instruments, gathered with observation, guided the decisions concerning the first dyads, which were the teacher/researcher's responsibility (César, 2024b). Upon analysing some classes, César (2013a) soon realized that pupils who participated in Cape Verdean culture, whose L1 was Creole, preferred a global approach to problems, spatial or geometrical reasoning, and a graphic representation solving strategy. For those participating in the dominant culture, whose L1 was Portuguese, their natural ways of approaching problems were step-by-step, using analytical reasoning, and arithmetic or algebraic solving strategies. But it was when she began a meta-analysis of the full IK's *empirical corpus* that she realized the importance of the L1 for mathematical performances, as claimed by César (2024b) and César and Machado (2024, submitted). César and Machado (2024) gave several examples regarding Tasks B and D, from the IACC. In this paper, the examples regard Tasks A and E. César (2024b) illuminated the importance of the personal performance shapers (PPS), particularly for those whose natural approaches, reasoning, and solving strategies are usually forbidden in Mathematics classes.

Task A (IACC)

Task A (Figure 1) assessed critical sense regarding mathematical information in news. As it referenced news from newspapers, it was rarely rejected. Hence it was chosen to be the first task of the IACC (César, 2024b; Machado, 2014). It mentions an imagined town (Fractopolis) where several robberies took place. The mathematical data in the news have many mistakes, that pupils were supposed to identify: (1) 1 in 20 is bigger than 1 in 25; (2) 1 in 20 is not 20%; and (3) the robberies increased, while the news mentioned they decreased.

Figure 1 – Task A – IACC

A. Comment, from the mathematical point of view, the following news:

Decreasing of the robberies in homes

From our correspondent in Fractopolis:

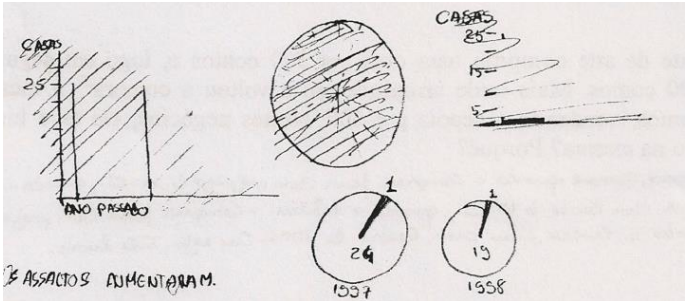
Police told that during last year, in Fractopolis, 1 in each 25 homes had been robbed. This year, the number of robbed homes changed to 1 in 20. So, there was a significant decrease but, even so, 20% is still a very high number. We hope that Fractopolis police will act in the right way to increase the safety of the citizens.

Pupils could solve Task A using any way of explaining their reasoning: calculations, drawings, diagrams, fractions, or others, but without using calculators. They knew that the IACC did not count for their school mark in Mathematics; it was intended to get to know them better, so that their teacher/researcher could choose the best practices according to their needs. Analysing all the solving strategies used, we realize that the number of pupils who elaborated complete explanations in successful solving strategies is identical for those who have Creole or Portuguese as their L1 (César, 2024b; César & Machado, 2024; submitted). Thus, the level of success is not shaped by their L1. What varies is the approach to problems, reasoning and solving strategies they prefer: a global approach, spatial and geometric reasoning, and graphic representation solving strategies, when the L1 is Creole; and a step-by-step approach, analytical reasoning, and arithmetic, algebraic or mathematical compositions as solving strategies, if L1 is Portuguese.

Some pupils realized that the number of robbed houses increased, but they did not explain their reasoning, as asked.

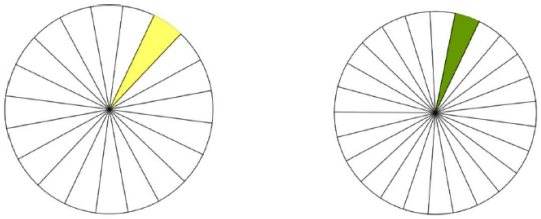
Figure 2 illustrates spatial reasoning and several graphic representation solving strategies (different attempts): this pupil drew a bar graph and a circular graph, which he crossed out; then, he used two circular graphs, with the part that represents the robberies, writing the years (that year - 1998 - and the previous one), which were not mentioned in the text, mobilizing concrete reasoning (CR). In the bottom left corner, he wrote: "the robberies increased". Thus, he recognized that some data were mistaken, but he did not clearly explain his reasoning.

Figure 2 – L1: Creole



Others remembered what happened if they ate pizza and understood that robberies had increased. Pizzas were the most common metaphor, but chocolates, lasagnas, and cakes were also used. The pupils who used food metaphors to solve this problem also showed that they were able to establish connections between school Mathematics and real-life Mathematics. Thus, they were able to do what Abreu, Bishop, and Presmeg (2002) called transitions between contexts of mathematical practices, an important issue when they begin their professional lives, and in their daily practices.

Figure 3 – L1: Creole (V.)



"Como mais se a pizza estiver dividida em 20 fatias"

V., a pupil whose life trajectory of participation was discussed by César (2009, 2013a, 2013b, 2024a, 2024b) and César and Machado (2024), used a global approach, geometric reasoning and a graphic representation solving strategy (drawing the pizzas, Figure 3). He drew the two pizzas, marked in colour the slice he ate, and wrote: "Pizza divided in 20 slices" (yellow) and "pizza divided in 25 slices" (green). Then he wrote: "I eat more if the pizza is divided in 20 slices". To give a complete explanation, the only thing missing is an answer in which he would go from the pizzas to what is mentioned in the text: the robberies. Other pupils gave these complete answers.

Figure 4 – L1: Portuguese

> The robberies have increased because if I eat a slice of pizza that was shared by twenty people, I eat more than if I eat a slice of the same pizza that was shared by twenty-five people.

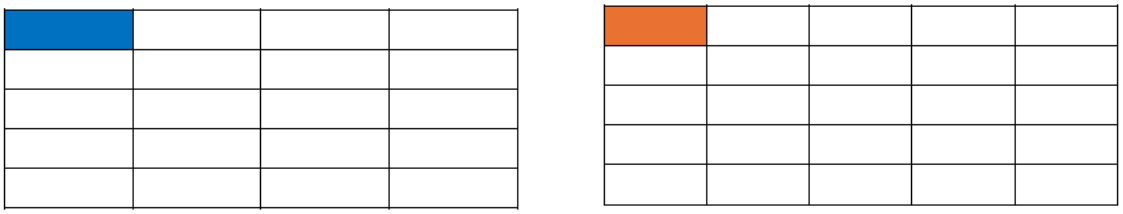
In Figure 4, the pupil preferred analytical reasoning and used a mathematical composition as solving strategy. These two examples illustrate that different reasoning and solving strategies can lead to a successful resolution, in which the

explanation is not complete. It is interesting to stress that, while in Figure 3 the pupil did not feel the need to say that the pizzas were the same size, as this can be seen in the drawing, when the solving strategy is a mathematical composition, the pupil stated that clearly.

Another pupil (Tatiana, whose dyad we discuss in this paper) also used geometric reasoning and a graphic representation solving strategy, this time using chocolates as a metaphor (Figure 5). Thus, she also establishes connections between school Mathematics and real-life Mathematics. As she did not think that "drawing was Mathematics" (Tatiana, IC, September 1999), she crossed out her solving strategy. We did it on the computer, repeating what she wrote on the board in the whole-class discussion (next day of classes), in which all pupils went to the board to explain a successful solving strategy (César, 2017; César & Machado, submitted). In the whole-class discussion, pupils were amazed when they realized there were different successful reasoning and solving strategies for each task. But they were also pleased, as they realized that their ways of reasoning and solving strategies would be accepted. Asking all pupils to explain a successful solving strategy was one of the first inter-empowerment mechanisms used by the teacher/researcher and had a very powerful implicit message: everyone can solve mathematical tasks in a successful way.

Tatiana wrote "chocolate divided in 20 pieces" (blue) and "chocolate divided in 25 pieces (orange). Then, she wrote "If two equal chocolates are divided in 20 and 25 little squares and I eat a little square of each one, I eat more in the one of the 20 little squares". Little squares (*quadradinhos*) it is the common term in Creole and in Portuguese for each small piece of chocolate, despite their shape, which is sometimes a square and other times a rectangle. Thus, she distinguished between squares and rectangles, but she used the everyday designation, and she mentioned this orally, in the whole-class discussion.

Figure 5 – L1: Creole (Tatiana)



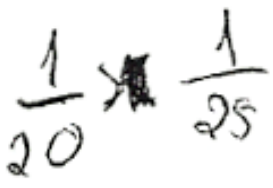
Chocolate dividido em 20 bocados

Chocolate dividido em 25 bocados

"Se dois chocolates iguais estiverem divididos em 20 e em 25 quadradinhos, e eu comer 1 quadradinho de cada, como mais no que tem 20 quadradinhos"

Some pupils realize that robberies have increased, but they do not explain their reasoning in detail, leaving the answer incomplete. They just write that $\frac{1}{20} > \frac{1}{25}$ (Figure 6). Answers that only mentioned that the robberies had increased, without any explanation, existed in any of the types of reasoning and solving strategies, as illustrated in Figures 2 to 6.

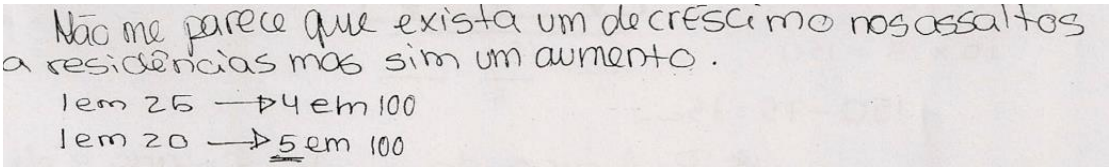
Figure 6 – L1: Portuguese



Others preferring analytical reasoning and an arithmetic solving strategy, gave complete answers (Figure 7). This pupil compares the percentage: "4 out of 100 and 5 out of 100" and wrote "I do not think that there is a decrease in the homes robberies, but an increase".

Some pupils whose L1 is Creole also elaborated complete explanations, because after drawing the pizzas, chocolates, or other metaphors, they wrote an answer, making the transition from the metaphor to the robberies. Thus, complete answers exist in a similar percentage for pupils from both L1: Creole and Portuguese.

Figure 7 – L1: Portuguese



The analysis of the reasoning and solving strategies used in Task A shows that pupils whose L1 is Creole prefer a spatial or geometric reasoning and a graphic representation solving strategy, while those whose L1 is Portuguese, choose analytical reasoning and an arithmetic or mathematical composition as solving strategies. César and Machado (2024) analysed examples of Tasks B and D, reporting similar results. César and Machado (submitted) discussed in a detailed way Tomás and Tatiana resolutions of the IACC, and César (2024b) did that for V. and M., coining a new concept: personal performance shapers (PPS).

For pupils whose L1 was Creole, who were used to seeing these resolutions discouraged or forbidden, it was very important that they were accepted (César, 2009, 2017, 2024b; César & Machado, 2024, submitted). Gradually, they went from peripheral to legitimate participants (César, 2009, 2013a, 2013b; César & Machado, submitted). The confidence they felt in realizing that different forms of resolution were legitimized helped to create thinking spaces (Perret-Clermont, 2004) and made the class begin to work as a learning community (Civil, 2007). For the first time, each pupil explained something that was relevant enough for their classmates to record in their notebooks. The whole-class discussion of the IACC took place in the second class of the school year. It had such an impact on the pupils, that they mentioned it quite often in interviews and informal conversations, even when many years had already gone by. The teacher/researchers also had several notes about this class in their diary, illustrating its importance for them and for the pupils.

Task E (IACC)

Task E was the last one, but pupils could choose the order in which they solved the five tasks. It evaluated the approach that pupils preferred to problems (global approach or step-by-step approach), and if they were able to connect school Mathematics and real life, particularly as regards purchases and sells, and the notion of profit and loss. The first version of this problem used *escudos*, the Portuguese currency until it was changed to euros, when Portugal joined the European Union (EU) in 2002. Thus, the text remained the same, but the currency was changed (Machado, 2014).

Figure 8 – Task E – IACC

E. An art dealer bought a painting for €300 and then he sold it for €400. Later, he regretted having sold it and so he bought it again for €500, selling it afterwards for €600. In these businesses, did he have a profit, loss or did he remain with the same money? How much was the profit or the loss if they existed?

In a similar way to what happened in the other tasks, the number of pupils who succeeded was similar for both L1 (Creole and Portuguese). But the language shaped which approach they preferred: (1) when L1 was Creole, they used a global approach, i.e., they summed up all the purchases, and all the sales, and then they calculated the profit (e.g., V. and Tatiana); and (2)

when L1 was Portuguese, they calculated each sum according to the order in which they were mentioned in the text, which corresponds to a step-by-step approach (e.g., M. and Tomás). These approaches were also observed when the pupils were solving other mathematical tasks in dyads, as illuminated by César (2009, 2013a, 2013b, 2024a, 2024b) or by César and Machado (2024). Teacher/researchers tried to put together, in the same dyad, pupils who preferred different approaches, reasoning, and solving strategies, as those who mobilize complementary abilities and competencies tend to progress more in their development and learning.

Since Task E refers to purchases and sales, pupils usually used analytical reasoning and an arithmetic solving strategy, no matter what their L1 was. This illustrates that reasoning and solving strategies are also shaped by the nature of the task. Thus, suggesting tasks that allow for a broader scope of reasoning and solving strategies is an important issue when we aim at promoting an intercultural and inclusive education (César, 2024b; César & Machado, 2024). Even if they are using an arithmetic solving strategy, those whose L1 is Creole often used a spatial disposition of the sums that seemed shaped by a graphic representation: they used arrows, and other symbols, and placed each sum in a way that was visually inspired in a graphic representation. Some resolutions also allowed to confirm if pupils mobilized concrete reasoning (CR) or abstract reasoning (AR), as illustrated in Figures 10 and 11.

Some pupils recognised that there was profit but they did not quantify it, as asked, so their resolution was incomplete. This happened in a similar percentage among those whose L1 was Creole or Portuguese. As they did not quantify the profit, a mathematical composition was often preferred as solving strategy (Figure 9). As the pupil in Figure 9 wrote: "He made a profit because he always sold the painting for more than he bought it." Thus, she used a step-by-step approach because she followed the order of the purchases and sales mentioned in the text. Figure 10 illustrates an incomplete resolution by a pupil whose L1 was Creole. She uses a global approach, summing the purchases and then the sales, which makes her solving strategy more complete than the one used in Figure 9. But she does not quantify the profit, as asked. She first wrote "This is what he spent" and then "This is what he earned". In her answer, she mentions: "I think he had a profit".

Figure 9 – L1: Portuguese

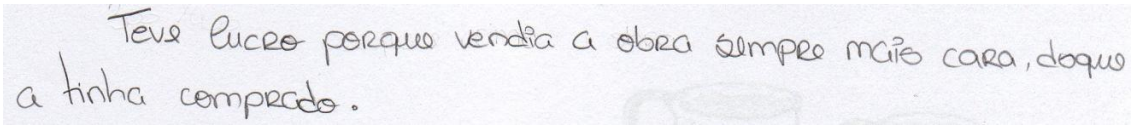
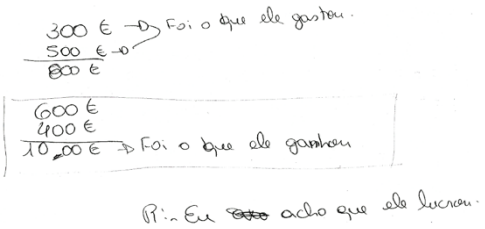
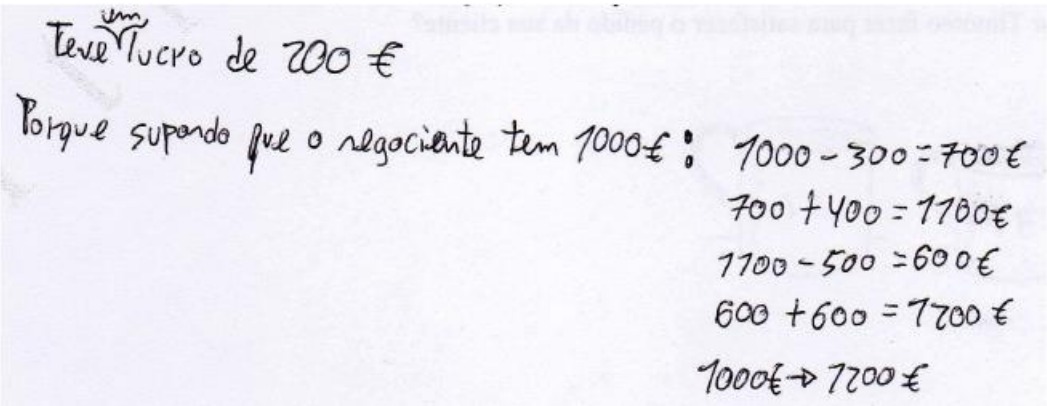


Figure 10 – L1: Creole



Other pupils needed to go back to a concrete situation and to imagine how much money the Art dealer had before the first purchase (Figure 11). Thus, while solving Task E, this pupil mobilized CR. He decides the Art dealer had 1000 Euros, and then he follows the order of the purchases and sales mentioned in the text. On the top he wrote: "He had a profit of €200", which means that this pupil wrote the answer before the sums corresponding to the transactions. Then he adds: "Because imagining the dealer has €1000" and after that he indicates all the sums.

Figure 11 – L1: Portuguese



Other pupils used a similar approach to the problem and solving strategy but mobilizing AR (Figure 12). Thus, this pupil operates with integers, both positive and negative. At the end, he writes the answer, quantifying the profit obtained: "He ended up winning €200".

Figure 12 – L1: Portuguese

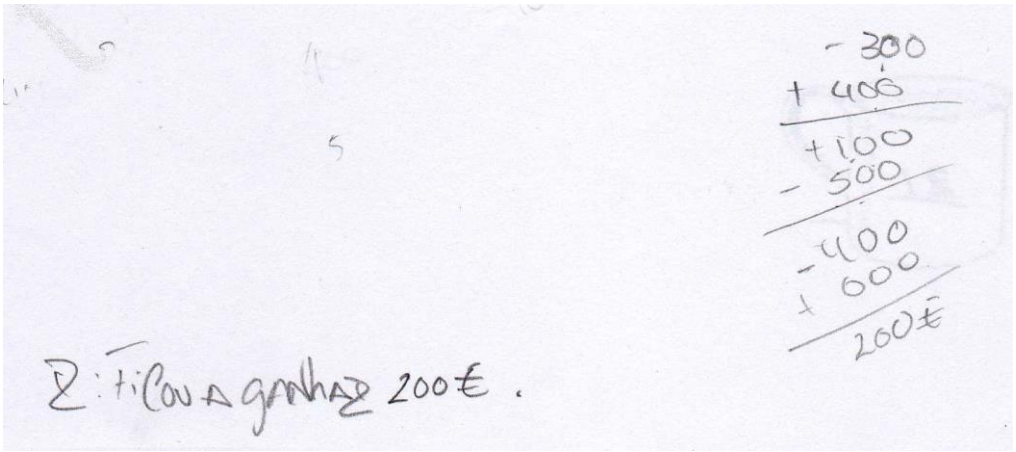
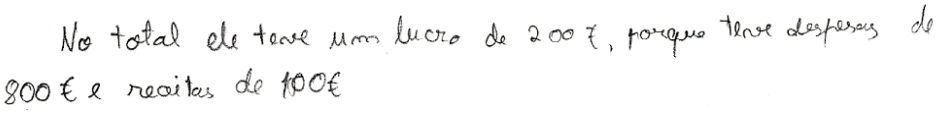


Figure 13 shows a very short but clear solving strategy, based on a mathematical composition. This girl, whose L1 is Creole, gave a complete answer. She used a global approach because she sums what he paid and what he received. Implicitly she also used an arithmetic solving strategy. She wrote: "In total he had a profit of €200 because he spent €800 and received €1000".

Figure 13 – L1: Creole



In Figure 14 a pupil whose L1 is Creole prefers a global approach and an arithmetic solving strategy. Although she did not write a final answer, the quantification of the profit is clear.

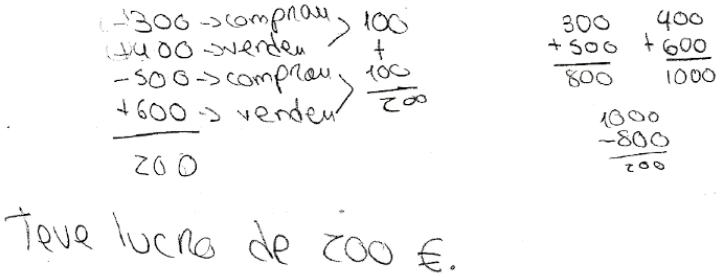
Figure 14 – L1: Creole

Purchases: 300 + 500 = 800€
Sales: 400 + 600 = 1000€

Profit: 1000 – 800 = 200€

Figure 15 shows a rare resolution: she uses two different approaches, which results in a very complete arithmetic solving strategy. On the left, she describes a step-by-step approach where the sums follow the order mentioned in the text. She uses a kind of sign to make clear that he had a profit of €100 each time he sold the painting, helping the quantification of the profit. On the right side, she uses a global approach and an arithmetic solving strategy. As these two approaches are shown in parallel, it is easy to realize the differences between them. She could also verify that these two approaches were successful as they both achieved the same profit. At the end there is an answer: "He had a profit of €200."

Figure 15 – L1: Creole



In Task E the approach that pupils prefer is shaped by L1. César (2024b) and César and Machado (2024, submitted) also showed that L1 shapes pupils' approaches, reasoning and solving strategies in other tasks from the IACC, as well as in the tasks they solved in dyads, when they studied new Mathematics contents. These results illuminate how important it is to allow pupils to use their natural ways of doing and learning Mathematics.

Tatiana and Tomás's dyad: Co-construction of solving strategies

The dyad whose performance and life trajectory of participation we analyse was made up of Tatiana, a girl who participated in Cape Verdean culture and whose L1 was Creole, and Tomás, a boy whose family was Portuguese and whose L1 was Portuguese. We have already discussed this dyad in previous papers (César, 2017; César & Machado, submitted). At that school around two thirds of the pupils were very poor, from a neighbourhood of slums with very few living conditions, where many people from Cape Verde and some very poor Portuguese lived (César, 2009, 2013b, 2017, 2024b; César & Machado, 2024, submitted). The other third lived in new middle class and upper-middle class blocks of flats and villas. When the IK project began working in this school (1994/95), these three groups (poor Cape Verdean, poor Portuguese, and well-to-do Portuguese) interacted very little and, if they did, it was in a confrontational manner. Thus, we began using regulatory dynamics of participation between the school and the families, reported in César (2013b). Later, the school used bi-univocal cultural mediation, a construct coined by César (2017). This type of mediation means that a team, with members from the different cultural groups, shares the decisions and evaluation of the practices regarding cultural mediation. Thus, while César (2009) and César and Machado (2024) describe a class (1997/98) in which spatial distribution of the pupils illuminates the conflicts and lack of communication, later this was no longer what we observed in the first week of classes.

Tatiana and Tomás's class participated in the IK project for three years: from 1999/2000 to 2001/2002, corresponding to their 7th to 9th grades (3rd cycle). As we wanted different cultures

to feel accepted and to promote intercultural dialogue, the first dyads were mixed in terms of culture, gender, abilities and competencies, and school marks, so that they could both work in his/her ZPD, taking turns as the more competent peer (César, 2009, 2024b; César & Machado, 2024, submitted). Both Tatiana and Tomás had Level 3 (the lowest positive level) in Mathematics in previous academic years. However, Tatiana had a negative social representation of herself as a Mathematics learner, while Tomás had a positive social representation of himself. Tatiana planned to leave school as soon as she completed the 9th grade and to start working. As she did not speak Portuguese when she began the 1st grade of primary school (she spoke Creole at home and in her neighbourhood), she had difficulties learning how to read and write in Portuguese. In Portugal, the law does not allow pupils to repeat the 1st grade, so she repeated the 2nd grade. Therefore, she was a year older than Tomás and, if she did not repeat another grade again, she would be 16 years old when she finished the 9th grade, the minimum age to be allowed to work. Tomás did not yet know what course he wanted in college, but he was sure of going to university. He really liked Sciences, but with Level 3 in Mathematics, he was afraid that he would not be able to get into the universities he preferred.

When their teacher/researcher decided to form this dyad, she was convinced that, as they had complementary abilities and competencies, they would progress quickly. Time proved her right: they both achieved Level 4 in the first individual test (October 1999). The IACC analysis had shown that Tomás had more knowledge of previous Mathematics contents, more perseverance in solving tasks and more resilience. But he had no critical sense in relation to mathematical information (in Task A he repeated what was written in the news, despite it being false), nor did he have mathematical intuition or creativity (Task B), something that Tatiana was able to mobilize. She preferred a global approach to problems (Task E), spatial or geometric reasoning and graphic representation solving strategies (Tasks A and D), while Tomás preferred a step-by-step approach (Task E), analytical reasoning and arithmetic, algebraic or mathematical compositions as solving strategies (Tasks B, D and E), as mentioned by César and Machado (submitted). Therefore, working as a dyad, they could develop abilities and competencies, as well as learn Mathematics.

In the second class in which they worked as a dyad their teacher/researcher suggested three problems as an introduction to equations. As this content is considered difficult, the teacher/researcher chose problems that allowed diverse solving strategies to be used. We analyse the social interaction while solving one of these problems. This dyad did not use an algebraic strategy (equations), but this happened in other dyads, enriching the whole-class discussion. In the interactive episodes TA means Tatiana and TO means Tomás.

Problem – A grocer sold half a cheese, then a quarter and finally a sixth. He then checked that 125g were left over. How many kilos did the cheese weigh in the first place?

1. TO – How do you think we can do it?
2. TA – I like drawing and solving like this. I don't even think it's Maths, but Miss seems to like it and even called me to the board on the chocolate one [whole-class discussion, Task A, Figure 5]. So, I'd draw the cheese and then mark what had been sold...
3. TO – As she likes different ways of doing it, we do both: yours, with drawings; and mine, with sums. This way, we even learn more and show that we worked hard. This also counts in the evaluation. Do you agree?
4. TA – Yeah. [She starts drawing a circle in her notebook and TO looks on attentively]
5. TO – Now you trace half…
6. TA – No, first I draw one sixth, which is the hardest…
7. TO – But first he sells half!...
8. TA – True. But he also sells $\frac{1}{4}$ and $\frac{1}{6}$. Tracing half and $\frac{1}{4}$ is easy. But tracing $\frac{1}{6}$ is more difficult to draw well… So, I'll start there…
9. TO – I disagree!
10. TA – I'll do it on this old sheet of paper, for you to see. [She draws a circle, marks one half and traces it, and does the same for $\frac{1}{4}$. TO nods his head, pleased. Then, she hands him the sheet and says:] Now, you trace $\frac{1}{6}$!
11. TO – Right, it's… it's… [He stops] I don't know where $\frac{1}{6}$ is…
12. TA - Well... if we start the other way around, we first trace $\frac{1}{6}$, then half and then $\frac{1}{4}$ [She draws as she speaks] and that way it becomes easy. [She looks at him, seeking his approval]
13. TO – I've got it. You can do this one on the answering sheet. You're real smart at drawing, much more than me!...

(Tomás and Tatiana, O, audio-recorded, September 1999)

Tomás was more confident than Tatiana. Thus, he assumed leadership, initiating the interaction (Talk 1). But he does that in a smooth way and respecting the didactic contract: he asks

Tatiana how she thinks they should solve this problem. Tatiana, who was shyer and more insecure, tells him that she likes to draw, but adds that she does not even think this is Mathematics (Talk 2). However, as the teacher/researcher called her to the board and validated her resolution in Task A (Figure 5), she decided to use a global approach, geometric reasoning, and a graphic representation solving strategy, in which she drew the cheese and what was sold. Tomás, who usually tries and knows how to please teachers, suggests that they solve the problem using Tatiana's solving strategy and his own, which is an arithmetic one. He justifies his suggestion: this way, they learn more (each one also learns the other's solving strategy) and it shows that they worked hard (Talk 3). His statement shows that he knows the evaluation criteria well and that pupil engagement counts. Participating in the dominant culture, as the son of parents with higher education degrees, he has no difficulty understanding the rules of the school. So, he uses them to his advantage or that of the dyad. Adhering to the didactic contract, he asks again if Tatiana agrees.

Then (Talk 4), Tatiana starts to draw the cheese and intends to mark, first, $\frac{1}{6}$ that was sold, which shows that she has highly developed geometric reasoning and that she uses a global approach to the problem, as she immediately visualizes each piece that was sold and how to draw them. Tomás, who prefers a step-by-step approach and who considers himself more successful in Mathematics than Tatiana, tries to get her to start by drawing one half, which the grocer sold first. In Talks 6 and 8, Tatiana tries to explain why they should start with $\frac{1}{6}$. But as she was unable to persuade Tomás, she decides to make a sketch, in the order he says, to prove to him that it does not work (Talk 10). At this point, Tatiana assumes leadership, and becomes the most competent peer. Thus, they are taking turns assuming that role, which favours development and learning (César, 2009, 2024b). Tatiana's decision works: Tomás tries to draw $\frac{1}{6}$ (Talk 11) and he is not able to do it. After that, Tatiana draws another circle (representing the cheese), starting with $\frac{1}{6}$ (Talk 12) and manages to mark each piece that was sold. So, in Talk 13, Tomás recognizes that, when it comes to drawing (the name he gives a graphic representation solving strategy), she has more developed abilities and competencies than him. Thus, he implicitly recognizes that she has more developed geometric reasoning and that he can learn from her.

14. TA – Now, look over here [She shows him the drawing of the cheese] I think the part that isn't traced is half of $\frac{1}{6}$, of the one I traced first...
15. TO – Hmm...hmm... [Affirmative sound] and we can check with the protractor... [He takes it, measures the angles... and smiles] You're a genius!... Just by looking, you immediately saw that it was half.
16. TA – And if it's half, it's $\frac{1}{12}$... it's as if you divided the cheese into 12 parts. So, that's what was left to sell, meaning $\frac{1}{12}$ is 125g. Now, just do the math...
17. TO – I've already done it here [He shows the calculator] It's 1.5kg. Write this on the sheet and give the answer.
18. TA – So this way is done. Now let's do yours.
19. TO – Mine is to add the fractions: $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}$. Do you know how to do it?
20. TA – We've got to put the same numbers here [She points to the denominators]
21. TO – Yes, to reduce it to the same denominator, which I think is 24. So, it's $\frac{12}{24}+\frac{6}{24}+\frac{4}{24}$, which is $\frac{22}{24}$, which is what he sold. The whole cheese is $\frac{24}{24}$. So, that leaves $\frac{2}{24}$ ... reducing it, you get $\frac{1}{12}$, the same result you got.
22. TA – So, then it's the same sums that we've already done. Write everything down and give the answer.
23. TO – We're way too smart! We did it in two ways and one checks the other, because they produce the same result. Miss is gonna be super happy.

(Tomás and Tatiana, O, audio-recorded, September 1999)

In Talks 14 and 16, Tatiana continues to show highly developed geometric reasoning, something that Tomás is unable to mobilize. The way she explains how she figured out that the part that was not sold is a half of $\frac{1}{6}$ (Talk 16) illuminates how developed her geometric reasoning is. It also illustrates how important it is to let pupils start with their natural ways of solving mathematical problems, as claimed by César (2009, 2024b) and César and Machado (2024, submitted). In Talk 15, Tomás makes affirmative sounds, but decides to confirm what Tatiana states by measuring with the protractor (which he had, and she did not, as she lived with financial difficulties and had hardly any school material). But he admires her ability to visualize and even calls her "genius". Thus, if we compare Tomás's reactions to those of M., discussed in César (2009, 2024a) and César & Machado (2024), we realize that they are quite different. Tomás, whose L1 is Portuguese, also preferred a step-by step approach to the problems, illustrated by his wish to mark what was sold in the same order mentioned in the text. He preferred analytical reasoning and an arithmetic solving strategy. But his natural way of solving this problem did not

mean he could not appreciate Tatiana's natural way of solving the problem. Thus, he praises her genuinely. M., whose L1 was also Portuguese, and who was a top achieving pupil, was unable to accept V.'s natural approach, reasoning and solving strategy. Thus, even when he elaborated impressive reasoning, or deduced rules for equations based on what he saw (César, 2024b), she only made tricky comments, that always had a negative tone. When we analyse V. and M.'s interactions, there is lack of trust, M.'s attempts to work individually, lack of will to learn V.'s geometrical reasoning and graphic representation solving strategies. This raised some conflicts that V. tried to avoid or prevent from escalating (César, 2009, 2024a, 2024b; César & Machado, 2024). Tomás also felt he was better in Mathematics than Tatiana, but he is genuinely interested in learning from her, he praises her geometrical reasoning, and they co-construct their solving strategies, because they help each other to progress in their resolutions. These differences illustrate two important issues: (1) peers that would not choose each other can react in quite different ways when they begin working together in dyads; and (2) even those who are more reactive, like M., realize after some weeks that they have learnt a lot, and become friends, as stated by César (2009, 2013a, 2013b, 2024a, 2024b) and César and Machado (2024).

Unaccustomed to being praised in Mathematics classes, Tatiana feels more confident and, even in the part where she needs to use an arithmetic solving strategy, realizes that what is left is 125g (Talk 16). Tomás, who feels comfortable with arithmetic solving strategies, takes the lead again and makes the final calculation, telling Tatiana to write everything on the sheet, including the answer. Once again, when remembering that she needs to write the answer, Tomás shows that he knows the usual rules of the didactic contract well, and that he wants a complete resolution of the problem. He wants their teacher/researcher to value their engagement. What we observe is that, despite using the approach, reasoning, and solving strategy preferred by Tatiana, this dyad was working in a collaborative and dialogical way, respecting the didactic contract, and learning from each other, assuming leadership and acting as the most competent peer in alternating moments.

Once Tatiana's solving strategy is finished (Talk 18), she suggests that Tomás explains his own. He prefers a step-by-step approach, analytical reasoning and an arithmetic solving strategy. In Talks 19 and 21, he solves the problem and takes the opportunity to confirm whether Tatiana knows how to add fractions (Talk 19). She does not remember the word 'denominator' (Talk 20), but she knows what they must do and learns this designation from Tomás (Talk 21), who, having Portuguese as his L1, memorizes scientific terms more easily. Pupils whose L1 is Creole use Portuguese only at school, and in some daily tasks, like when they go to shops. As we do not use scientific names, like denominator, in those daily tasks, they need an extra effort to memorize them and to be able to mobilize them. Thus, working in dyads with Portuguese peers also helps them to master specific mathematical designations. In Talk 22, it is Tatiana who tells him to write everything down and give the answer, revealing a hint of leadership, a rare thing for her at the beginning of the school year. In Talk 23, Tomás makes an interesting statement: using two different solving strategies allows them to check whether they are both successful.

This peer interaction corroborates what we had observed when they were solving the IACC tasks: those whose L1 is Creole prefer global approaches, spatial or geometric reasoning and graphic representation solving strategies. The ones whose L1 is Portuguese prefer step-by-step approaches, analytical reasoning and arithmetic or algebraic solving strategies (César, 2024b; César & Machado, 2024, submitted). This dyad resolution highlights the importance of a different first week of classes, in which pupils feel free to use their natural ways of doing Mathematics, and then learn others. But it also illustrates that if pupils work in dyads or small groups, with peers that have complementary abilities and competencies, they learn more, because they use different approaches, reasoning, and solving strategies when they solve mathematical tasks. These mathematical experiences have positive impacts on self-esteem, mathematical performances, school achievement, and in the relationships between different cultures. It illustrates how collaborative and dialogical work, namely in dyad, promotes the pace of work and commitment to solving mathematical tasks. Each class has only one teacher. But if each pupil has a classmate to work with, with whom to clarify his/her initial doubts, then the teacher has more time to interact only with those who were unable to overcome some difficulty. The existence of several reasoning and solving strategies also makes the whole-class discussion richer, and a fine opportunity to learn more.

Working in the ZPD (Vygotsky, 1934/1962), and acting in turns as the most competent peer (César, 2009, 2013b, 2024a, 2024b; César & Machado, 2024), each pupil transforms potential development into real development. In these excerpts, Tomás is developing geometric reasoning and visualization abilities, while Tatiana is developing scientific vocabulary (e.g., denominator) and analytical reasoning associated with arithmetic strategies, such as adding fractions. We see that a thinking space (Perret-Clermont, 2004) and a learning community (Civil, 2007) have been created, allowing pupils to express doubts, questions and discuss different points of view. In informal conversations and interviews, both Tomás and Tatiana stated that they had progressed by working as a dyad, although they recognize that would not have chosen the other as his/her peer. Tatiana illustrates that when she states that "If I could choose, I'd be with a girl I've known since primary school, who is my friend and lives near me. She's my best

friend." (Tatiana, I1, December 1999). Tomás told us something similar: "I would choose to work with my best friend, because sometimes we study together, at my home or his" (Tomás, I1, December 1999). Thus, the main criterion, if they could choose, would be friendship. But that would mean working with peers from the same gender, culture, and who mobilized similar approaches, reasoning, and solving strategies. Being such good friends probably would mean that they would have a lot more to talk about than Mathematics. Thus, these preferred peers would not contribute to increase learning, development, and intercultural dialogue. In the 7th grade, pupils usually choose to work with same-gender peers. Thus, these dyads would not contribute to decrease gender negative discrimination. As the IK project had other criteria to form the dyads, they usually were not the ones pupils would form.

But by the end of the 1st term (mid-September to mid-December), pupils had already realized how beneficial it was to work in their first dyads. Thus, when dyads were changed (November), to avoid dependence (e.g., thinking they were only successful if they worked with that peer), to promote more development and knowledge by interacting with another peer, and to promote socialization, pupils were usually sad to stop working with that first peer. Tomás expressed that in his first interview and illustrated how important it is to interact with different classmates, to promote socialization. He evens recognizes that knowing more classmates better creates a different class atmosphere, contributing to what the IK team called intercultural dialogue.

> In November, I just felt like going on working with Tatiana. We worked so well together, she's so smart, and we had such a good mark in the first test [Level 4]. But I must confess that I also learnt a lot with my second peer, and I also loved working with her. So, now, I'm no longer afraid of changing to another peer, and I realized that I became closer to them, I know them better, and they know me, and it's important to know more classmates well. The class has a much better and easy-going atmosphere nowadays. (Tomás, I1, December 1999)

Tomás reported that, since working with Tatiana, he had met more people from the Cape Verdean culture and that this had made him feel safer when he left school at night, when the days were shorter, as he started to feel protected and appreciated by those classmates and their friends and families. Collaborative and dialogical work in class proved to be an important tool to increase an intercultural dialogue. It also contributed to respect cultural diversity, promoting equity and social justice in Mathematics classes. Thus, it was part of the bi-univocal cultural mediation (César, 2017; César & Machado, 2024, submitted), used in this school, in which participants from each culture worked in a team that handled cultural mediation decisions, practices, and evaluation of those practices.

Tatiana, who began the school year with a negative social representation of herself, as a Mathematics learner, feeling discriminated for being from the slum and black, stressed other important issues. She became more confident, got higher marks, felt that her natural ways of doing and learning Mathematics were respected and valued by her teacher/researcher and her classmates, and that meant a major change in her school life and experiences. She overcame her doubts about her abilities and competencies, and for the first time she experienced an intercultural and inclusive education.

> When I realized that I could do Mathematics in my own way, that was a surprise, but it was also a relief. I felt respected and grateful because I could think and be as I am. This had an impact I didn't believe at first: I had Level 4 in the first individual test, in October. I never thought this was possible. Then, I was afraid when I changed to another peer. I was convinced I could only learn with Tomás. I thought the next peer wouldn't like working with me. But he did. That was amazing. Rich and white classmates were fond of me, of my ways of doing Mathematics. I felt I had a place in school, like I never did before. (Tatiana, I1, December 1999)

Tomás wished to take a Science course. To do this, he needed to get higher marks in Mathematics. This class participated in the IK project for three years. He managed to achieve Level 5 (the highest) in Mathematics, in the 9th grade, and have high marks in secondary school (10th to 12th grades), the averages of which count towards college admission. This expanded the possibilities of choosing courses, having opted for dentistry, as explained in detail by César and Machado (submitted). He completed his university studies, as he told us in the follow-up, and then he began working. Now he has his own dentist's office, and he feels proud of his career. Thus, for him and his family, becoming a top achiever at school was a source of pleasure. It created no conflicts between his I- and Me-positions (César, 2017; César & Machado, submitted).

Tatiana improved her academic performance a lot, but this caused her some conflicts between her different I-positions (e.g., pupil and daughter; classmate and neighbour), as her family wanted her to start working and have children at a very young age, and she decided to go on studying. In the interviews and several autonomous informal conversations during the follow-up and when this was formally over, she reported this suffering, but also the resilience she had already developed

(César, 2017; César & Machado, submitted). Her academic success made her question the life trajectory of participation she wanted to trace, choosing to enrol in secondary school, even against her family's opinion. To achieve this, she had to start working part-time, with a very busy schedule. After the 12th grade, she worked for a year to save money to rent a room and take a nursing course and, later, specialize in neurosurgery (César, 2017; César & Machado, submitted). Thus, for them both, collaborative and dialogic work had strong impacts on their lives, both at school and outside it. But, for Tatiana, her life trajectory of participation involved many more struggles, and overcoming countless barriers.

Final Remarks

The ways in which power is used are sometimes subtle. Often, the implicit says what has not been made explicit. Therefore, it is not easy for pupils to recognise the forms of power to which they are subjected, nor to understand how they can overcome these barriers. The fewer intra-empowerment mechanisms they have internalized, the less voice and agency they feel they have, and the more difficult it is to move from peripheral participants to legitimate participants (César, 2009, 2013a, 2013b, 2017, 2024b; César & Machado, submitted). The systems themselves (educational, social, justice) do not facilitate the empowerment of vulnerable minorities (César, 2009, 2013a, 2017; César & Machado, submitted). Therefore, to achieve an inclusive society and education, a lot of work needs to be done, particularly in Mathematics classes. Ernest (2024) discussed several issues regarding Mathematics Education, in these challenging intercultural societies and schools.

> But we must acknowledge that the good can occur at several levels in education and that the good is itself contingent, therefore depending on time, place and scale. Are the aims of teaching mathematics wholly good? Are the planned curricula for mathematics relevant, appropriate and beneficial for all? Are the teaching practices and pedagogies optimal for all children and successful for all students? What do the outcomes of mathematical schooling tell us about the spread of benefits for learners? Are the assessment and certification systems used for mathematics wholly beneficial and in no way harmful? Is mathematics education good for everyone or is it less good or even damaging for some? (Ernest, 2024, p. 4)

These are essential questions to address in teachers' pre- and in-service education, and in researchers' teams, to develop their epistemological conscience (César, 2010; César & Machado, submitted), a major issue when we aim at achieving an intercultural and inclusive education and research. The examples we analysed illuminate how different L1s shape different approaches to problems, mathematical reasoning and solving strategies. Accepting only those associated to the language of instruction and the dominant culture means negatively discriminating pupils who participate in vulnerable minority cultures, such as those who speak Creole (Cape Verde). César (2009, 2017, 2024b) and César and Machado (2024, submitted) show that this rejection of natural ways of solving mathematical tasks is felt as an aggression, as a form of violence. Ernest (2024), reflecting about the ways to increase human dignity, also assumes that "each individual's uniqueness" must be respected, and that "everyone has a right to a voice", two core issues in an intercultural and inclusive (Mathematics) education.

> Making sure that everybody has equal opportunities and equal rights is not enough to ensure human dignity. It also requires respecting each individual's uniqueness, needs and autonomy. Dignity requires acknowledging that everyone has a right to a voice, and that their particular narratives of self, their opinions, desires and choices are listened to and responded to as coming from an equal. (Ernest, 2024, p. 5)

Dulce Maria Cardoso (2024), a Portuguese writer, said in an interview: "Power is imposed through violence and submission, and this is done through the acceptance of violence. Violence is becoming natural. People see it, but it takes them many years to realize that it is violence." (no page). Although she is referring to another context (returnees), it portrays what happens in Mathematics classes, regarding pupils participating in the Cape Verdean culture. Violence has become a common practice by rejecting their natural ways of approaching mathematical tasks, by thinking that it is expected they will underachieve and experience early school dropout. It is urgent to realize that not accepting their natural ways of doing and learning Mathematics corresponds to using violence against them and discriminating against them in a negative way. Most teachers ignore how L1 shapes mathematical performances. Thus, they do not even realize they are acting in a violent way when they forbid these pupils from using their natural ways of reasoning. They do not realize that this means forcing these pupils to perform below their abilities and competencies, to experience short and painful paths at school.

The IK data analysis illuminated the importance of changing the first week of classes and of knowing pupils' abilities and competencies in a timely manner (César, 2009, 2017, 2024b; César et al., 2014; César & Machado, 2024, submitted). Designing this first week in a way that conveys implicit empowering messages is important for all pupils. But it is

particularly relevant for those participating in vulnerable minorities, who tend to have short educational paths. This first week is essential to develop thinking spaces (Perret-Clermont, 2004) and learning communities (Civil, 2007), based on collaborative and dialogic work, promoting pupils' development and learning. Mathematics education can be a vehicle for empowerment, helping to respect and celebrate different cultures (César, 2009, 2013b, 2017, 2024a, 2024b; César & Machado, 2024, submitted). To achieve this, the curriculum and classroom practices must be used as a vehicle for inclusion (César, 2009, 2013a, 2024b; Rose, 2002) and not as (another) form of exclusion.

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