Global Academic and Scientific Journal of Multidisciplinary Studies (GASJMS)

Volume 3 | Issue 3, 2025 Journal Homepage: https://gaspublishers.com/gasjms/ ISSN: 2583-8970
Email: gaspublishers@gmail.com

Probability Density Function Estimation Using a New Series Nonparametric Estimator with Data Applications

Ejakpovi, S. U.[1]; Siloko, I. U.[2]; Ojobor, S. A.[3]; Ishiekwene, C. C.[4]

[1]Department of Mathematics, College of Education, Warri, Delta State, Nigeria
[2]Department of Mathematics, Edo State University, Iyamho, Edo State, Nigeria.
[3]Department of Mathematics, Delta State University of Science and Technology, Ozoro.
[4]Department of Statistics, University of Benin, Benin City, Edo State, Nigeria.

Received: 20.05.2025 | **Accepted:** 25.05.2025 | **Published:** 04.06.2025

***Corresponding Author:** Ejakpovi, S. U.

DOI: *10.5281/zenodo.15588414*

Abstract **Original Research Article**

In this paper, we studied a new nonparametric density estimation method. We proposed this estimator in series univariate form that incorporates the kernel density estimator and kernel density differential estimator and derived its normality and variance. We have also compared the asymptotic normality of the mean integrated squared error (AMISE) of the proposed estimator, kernel density estimator and bias reduced kernel estimator. The results obtained have shown that the proposed estimator, Hermite series kernel density estimator one has better performance than the kernel density estimator and bias reduced kernel estimator when real data are applied.

Keywords: *Hermite Series Kernel Density Estimator One, Kernel Density Estimator, Bias Reduced Kernel Estimator and Asymptotic Mean Integrated Squared Error (AMISE).*

Citation: Ejakpovi, S. U., Siloko, I. U., Ojobor, S. A., & Ishiekwene, C. C. (2025). Probability density function estimation using a new series nonparametric estimator with data applications. *Global Academic and Scientific Journal of Multidisciplinary Studies (GASJMS)*, *3*(3), 91-98.

1. INTRODUCTION

Nonparametric density estimation is advancing in many fields of study as its applicability is evidential in its strides in the Mathematical Sciences, Statistics, hydrology and so on due to no pre-specified functional form for the assumed distribution of data. This density estimation differs from the parametric estimation that adopts any of Gaussian, Gamma, Cauchy as the underlying functional distributional form for the data with key assumptions. The parametric estimation techniques include the Bayesian parametric estimation and maximum likelihood estimation and has validated an extra ordinary volume in dealing with both high dimensional data such as images, sound or videos and large data set (Lawal et al., 2024). The nonparametric estimation violates the assumptions on the data, thereby allowing the data speak to for themselves with the merit of being easier to understand which allows linear memory and sub linear query time for density estimation (Tosatto et al., 2021).

Some notable methods like histogram, orthogonal series estimators, restricted maximum likelihood estimator and host of others. In literatures, among these methods, the easiest and most popular is the histogram, which seems so easy to adopt at first instance most especially for identifying data distribution processes. Due to its limitations in roughness, it has been replaced by other different nonparametric methods like the kernel density estimator, which is the basis of kernel smoothing techniques in nonparametric estimations (Fleming & Calabrese, 2017). The kernel density estimator has gained tremendous attention due the influence of computer software programs which allow it to be performed on data set of ten million samples in implementing its structures and its adoption depends on the expected complexity of the distribution and dimensionality of the data.

The kernel density estimation is accredited for its data visualization and exploration which gives concise information about data under investigation (Ziane et al., 2021; Humbert et al., 2022). It produces a smooth empirical probability density function (pdf) estimates which seems to be better represent the true probability density function of a continuous variable (Weglarcyzk, 2018). The univariate dimensional representation of the kernel density estimator in its concise form is:

$$\hat{f}(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{h_X}K\left(\frac{X_i - x}{h_X}\right) \qquad (1)$$

where $h_X > 0$ is the smoothing parameter, n is the sample size, X_i are the sets of observations and x is at which value the function is estimated and $K(.)$ is the kernel function which is non-negative, symmetric and usually a probability density function required to satisfy the axioms:

$$\int K(x)dx = 1, \int xK(x)dx = 0 \ and \ \int x^2K(x)dx < \mu_2(K) \qquad (2)$$

These axioms confirmed the kernel functions descriptive properties that help to determines the estimator's rate of convergence. The general form of this symmetric kernel family is:

$$K_p[u] = \frac{\left(\frac{1}{2}\right)_{p+1}}{p!}(1-u^2)^p \qquad (3)$$

where $p = 0, 1, 2, \dots, \infty$ is the polynomial index and its values produce Uniform, Epanechnikov, Biweight kernels respectively (Ejakpovi & Ozobokeme, 2024). And when the polynomial index goes to infinity, it produces the Gaussian kernel function that exists in one, two and n-dimensions respectively with σ as the inner scale that determines the spread of the Gaussian kernel function (Sha & Xie, 2016). The kernel density estimator turns each data points into a smooth Gaussian bump and sum up all of these bumps to get a smooth estimated pdf. This estimate is asymptotically equal to the actual probability density function (pdf) f plus some errors which decrease with large values of the sample sizes, n. It is a suitable choice to trade performance determined by the mean squared error that decrease by $n^{-4/5}$ with the crucial role of the smoothing parameter, h_X (Hang et al., 2018).

The performance of the kernel density estimator is mesh independent because of the point wise convergence for literatures on kernel density estimator's performance are on the increase. This necessitated the development of fast converging kernel estimators as a topical research in nonparametric estimation. And some of these estimators include the average singular integral estimator (Delgado & Vildal-Sang, 2002), two new Kernel estimator (Rodchuen & Suwatte, 2011) and bias reduced kernel estimator (Xie & Wu, 2014). Therefore, this paper is organized in the following sections: in section 2, we shall propose a new estimator, in section 3, we shall obtain the performance metrics of the proposed estimator, in section 4 we shall discuss results and conclude the study in section 5.

2. METHODS AND MATERIALS

The need of improving the performance of kernel estimators has led to several approaches of which the reduction estimators have been mostly considered methods. The method stems from the rth derivative of $\hat{f}(x)$ which is expressed as:

$$\hat{f}^{(r)}(x) = \frac{1}{nh^{r+1}}\sum_{i=1}^{n}K^{(r)}\left(\frac{X_i - x}{h}\right) \qquad (4)$$

where $K^r = (-1)^r H_r(\mathrm{x})K(\mathrm{x})$ is the r^{th} derivative order of the kernel function with at least r non-zero derivatives and incorporates the Hermite Series Polynomial of r^{th} derivatives orders (Raykar et al., 2010), (Raykar et al., 2015) and (Mynbaev et al., 2015). The Mathematical formulation of the propose kernel estimator is:

$$\begin{cases} \widetilde{m}_n(x) = \hat{f}(x) - asymptotic\ Bias\left(\hat{f}^s(x)\right) \\ \qquad = \frac{1}{nh_X}\sum_{i=1}^{n}K\left(\frac{X_i - x}{h_X}\right) - \frac{h_X^2 I_2}{2!}\hat{f}^{(s+2)}(x) \end{cases} \qquad (5)$$

where $s = 0, 1, 2, \dots$, h_X is the smoothing parameter and $\hat{f}^{(s+2)}(x)$ is the $(s+2)$ times continuously derivatives of the distributional function. When the value of the differential order is fixed, taking $s = 0$ in Equation (5) yield $\hat{f}^{(2)}(x)$ whose value is obtain from Equation (4) with the Hermite polynomial, $H_2(x) = x^2 - 1$. Thereafter, substitute its expression into Equation (5), generates the new nonparametric density estimator, called the Hermite Series Kernel Density Estimator One ($HSeKDE\ 1$) as:

$$\tilde{m}_n(x) = \frac{1}{nh_X}\sum_{i=1}^{n} K\left(\frac{X_i - x}{h_X}\right)\left[1 - \frac{I_2}{2}\left(\left\{\frac{X_i - x}{h_X}\right\}^2 - 1\right)\right] \quad (6)$$

where $I_2 = \int x^2 K(x)dx$, n is the sample size of the random variable X, x is at which value the function is estimated.

3. PERFORMANCE METRICS

The performance of any kernel density models takes several evaluation metrics, however the ease of Mathematical tractability, the global error criterion function called the asymptotic mean integrated squared error ($AMISE$) shall be deduce for the Hermite Series Kernel Density Estimator One ($HSeKDE\ 1$) in Equation (6) and shown to improves the Bias and variance of the estimator. So, suppose f is an unknown density function and kernel function satisfying the following axioms:

$$\begin{cases} \|f''\|_2^2 = \int_{-\infty}^{\infty} (f''(x))^2 dx < \infty \\ \|K\|_2^2 = \int_{-\infty}^{\infty} K^2(x)dx < \infty \\ \int_{-\infty}^{\infty} xK(x)dx = 0 \\ I_2 = \int_{-\infty}^{\infty} x^2K(x)dx < \infty \\ f'' \text{ is absolutely continuous in } x \end{cases} \quad (7)$$

Theorem 1: Under the axiom on f, K and h_X for the estimate of $\tilde{m}_n(x)$ in Equation (6), the bias and variance of the Hermite Series Kernel Density Estimator One ($HSeKDE\ 1$) are:

$$\text{Bias}(\tilde{m}_n(x)) = \frac{h^2 I_2}{2} f''(x) - \frac{I_2^2}{2} f(x) + \frac{I_2}{2} f(x) + \frac{h^2 I_2^2}{4} f''(x) + O(h^2)$$

and

$$Var(\tilde{m}_n(x)) = \frac{1}{nh} f(x) \int K^2(t)dt + \frac{I_2}{2nh} f(x) \int K^2(t)dt + O\left(\frac{1}{nh}\right)$$

Hence, a measure of the global accuracy of $\tilde{m}_n(x)$ is the mean integrated squared error ($MISE$) and since $f(x)$ is a probability function which is integrated over x will result to the asymptotic mean integrated squared error ($AMISE$), given as:

$$AMISE(\tilde{m}_n(x)) = \frac{h^4 I_2^2}{4}\|f''\|_2^2 - \frac{h^2 I_2^3}{2}\gamma + \frac{h^2 I_2^2}{2}\gamma + \frac{h^4 I_2^3}{4}\|f''\|_2^2 + \frac{I_2^4}{4} + \frac{h^2 I_2^3}{4}\gamma - \frac{I_2^3}{2} - \frac{h^2 I_2^4}{4}\gamma + \frac{I_2^2}{4} + \frac{h^4 I_2^4}{16}\|f''\|_2^2 + \frac{1}{nh}\|K\|_2^2 + \frac{I_2}{2nh}\|K\|_2^2 \quad (8)$$

where $\gamma = \int f''(x)dx$. On the basis of theorem 1, the asymptotic optimal smoothing parameter that minimizes the $AMISE(\tilde{m}_n(x))$ is:

$$h_{opt} = \left(\frac{2(\|K\|_2^2)^2}{(I_2^2\|f''\|_2^2)^3}\right)^{\frac{1}{13}} n^{\frac{-2}{13}} \quad (9)$$

The choice of the optimal smoothing parameter, h_{opt} of h_X is possible when all the quantities are all known in real data situation. Thus, the asymptotic optimal smoothing parameter in Equation (9) depends on the second derivatives of the underlying distribution which will be replace with an estimate and has an order, $O\left(n^{\frac{-2}{13}}\right)$.

4. RESULTS AND DISCUSSIONS

The proposed Hermite Series Kernel Density Estimator One ($HSeKDE$ 1) is used in the visualization and exploration of two data set of which the asymptotic mean integrated squared error ($AMISE$) values of these data were computed using Equation (6) on the platform of Mathematica version 12.3 software**.** The first data is the amount of Revenue Spent by fifty (50) companies on Rural Development (McClave and Benson, 1988). The kernel estimates of revenue spent depicts the amount spent on rural development. The kernel estimates indicated that the data were unimodal with the application of the Bias reduced kernel estimator ($BRKE$) and the kernel density estimator (KDE). The mode is observed to be lying between 6 and 8 with the peak region at 8. However, the proposed Hermite Series Kernel density estimator one $HSeKDE$ 1 shows that the data is bimodal with the mode occurring at 7 and 13. The regions of peaks of the data represented the most funds spent on rural development. The probabilities of the peaks of the various estimates lie between 0 and 0.15. The graphs below show the density estimates with selected kernels.

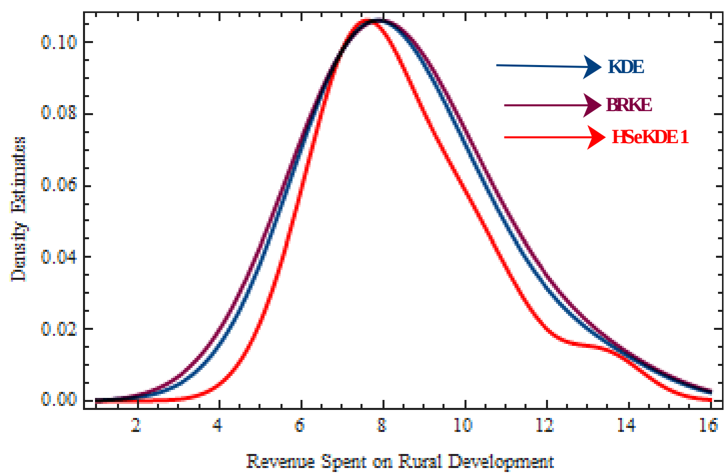

Fig. 1: Epanechnikov Kernel Estimates of Revenue Spent on Rural Dev. Data

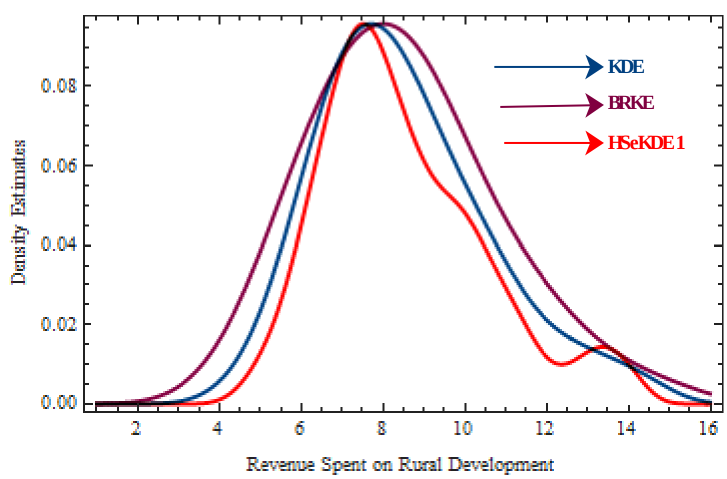

Fig. 2: Biweight Kernel Estimates of Revenue Spent on Rural Dev. Data

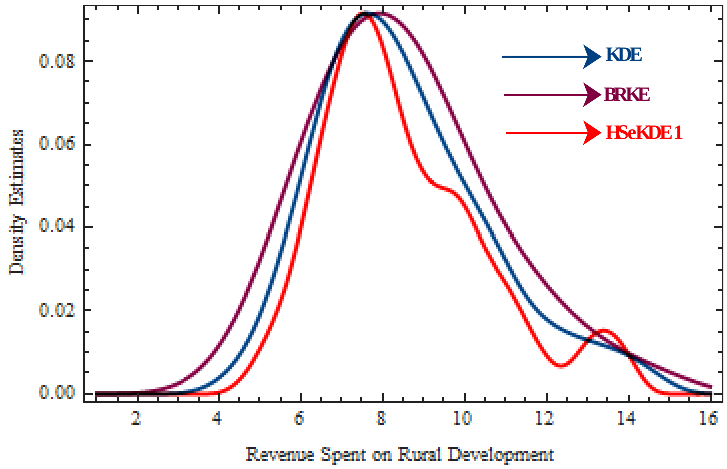

Fig. 3: Triweight Kernel Estimates of Revenue Spent on Rural Dev. Data

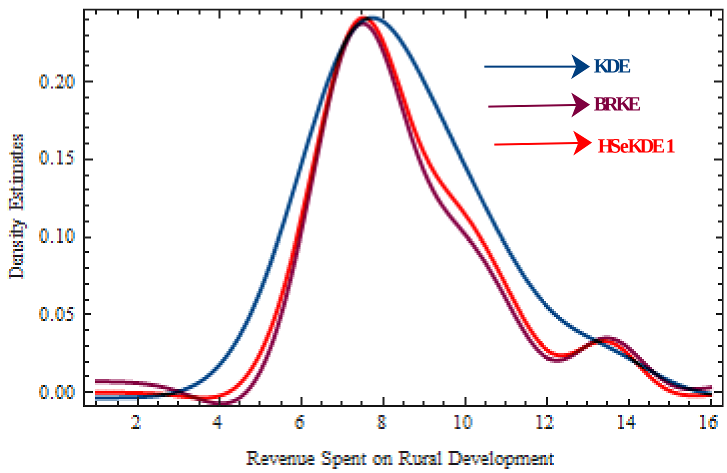

Fig. 4: Gaussian Kernel Estimates of Revenue Spent on Rural Dev. Data

The kernel estimates modality features of the observation data as shown in Figures 1, 2, 3, and 4 with the Epanechnikov, Biweight, Triweight and Gaussian kernel estimates of revenue spent on rural development by the estimators. The Figure 4 being the kernel estimates of the data with the Gaussian kernel function and the optimal smoothing parameters in this case are 1.20405, 1.09171 and 0.98804 using the KDE, $BRKE$ and $HSekDE$ 1 respectively. The unimodal property of the data by the KDE shows that the highest revenue spent is 8 dollars on rural development while both the $BRKE$ and $HSeKDE$ 1 depict the bimodal property of the data with regions of highest revenue spent as 7 and 13.5 dollars on rural development. Table 1 gives the smoothing parameters values, Statistical qualities and the measure of performance values of these kernel estimators.

Table 1: ***Kernel Estimators Smoothing Parameters, Statistical Qualities and AMISE Values***

Estimators	Kernels	$R(K)$	$\mu_2(K)$	h_{opt}	$\int Bias^2(\tilde{m}_n(x))dx$	$\int Var(\tilde{m}_n(x))dx$	$AMISE$
		Revenue Spent; Sample Size, $n = 50$.					
KDE	Epan.	0.60000	0.20000	2.12008	1.54949E-03	5.52571E-03	7.07520E-03
	Biweight	0.71429	0.14285	2.51159	1.49553E-03	5.61427E-03	7.10980E-03
	Triweight	0.81585	0.11111	2.85203	1.46089E-03	5.69057E-03	7.15146E-03
	Gaussian	$1/\sqrt{2\pi}$	1.00000	1.20405	3.68020E-03	1.47208E-02	1.84010E-02
$BRKE$	Epan.	0.60000	0.20000	2.38788	3.64353E-07	8.909220E-04	8.91286E-04
	Biweight	0.71429	0.14285	3.67973	2.17976E-07	5.218810E-03	5.21903E-03
	Triweight	0.81585	0.11111	4.29248	7.55895E-08	5.399070E-03	5.39915E-03
	Gaussian	$1/\sqrt{2\pi}$	1.00000	1.09171	5.05843E-04	4.046750E-03	4.55259E-03

$HSeKDE$ 1	Epan. Biweight Triweight Gaussian	0.60000 0.71429 0.81585 $1/\sqrt{2\pi}$	0.20000 0.14285 0.11111 1.00000	1.39010 1.52264 1.63013 0.98804	7.13991E-08 2.97503E-08 1.55068E-08 1.98779E-07	1.15722E-06 2.24865E-05 2.89614E-05 4.94603E-05	1.22862E-06 2.25165E-05 2.89614E-05 4.96591E-05

Generally, the performance of any kernel estimator is dependent of the choice of the smoothing parameters and least value of the asymptotic mean integrated squared error ($AMISE$) (Siloko et al., 2023) and (Yang, 2023). It can be observed from Table 1 that as the power of the kernels increases, the smoothing parameter and $AMISE$ values are also on the increase with the $HSeKDE$ 1 having the minimal values in these parameters. The computed $AMISE$ values of the $HSeKDE$ 1 clearly shows it outperformed the other kernel estimators and it achieved the smallest value of precision error of 0.0000496591 with the Gaussian kernel.

The second data is the ages at marriage for one hundred (100) women that applied for marriage licenses in Cumberland County, Pennsylvania USA (Sabine and Brian, 2004). The kernel estimates of ages at marriage of women is the years in time of their marriage. The kernel estimates indicated that the data have multimodality property through the visualizations displayed by the three kernel estimators. The peak of women marital age lies between 20 and 28 years of the women's age and revealed the age gap when marriages are usually more. The probability of getting married at these ages is high for the men but it tends to reduce between ages 30 and 40. A careful investigation of the kernel estimates also shows that between ages 42 and 58, there is probability of contracting marriage perhaps for widows or late decision makers. The probabilities of the peaks estimate lie between 0 and 0.038. The graphs below show the density estimates with selected kernels.

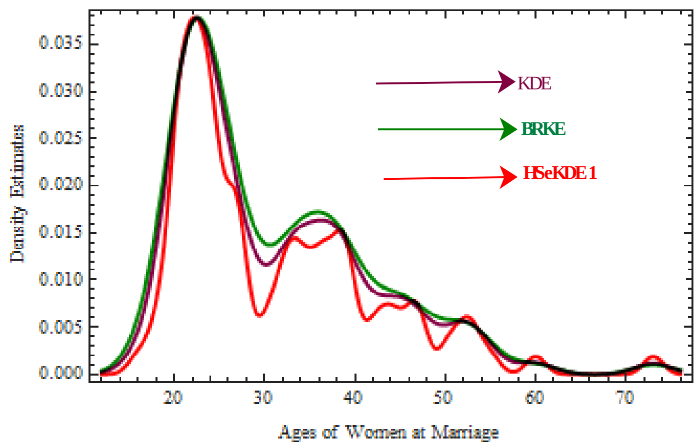

Fig. 5: Epanechnikov Kernel Estimates of Ages of Women at Marriage Time Data

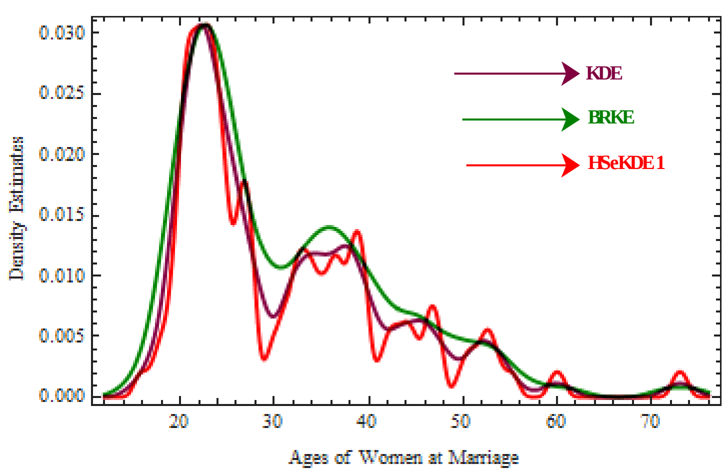

Fig. 6: Biweight Kernel Estimates of Ages of Women at Marriage Time Data

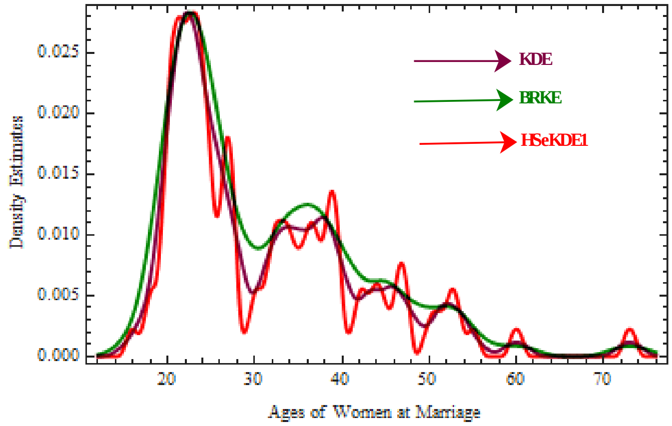

Fig. 7: Triweight Kernel Estimates of Ages of Women at Marriage Time Data

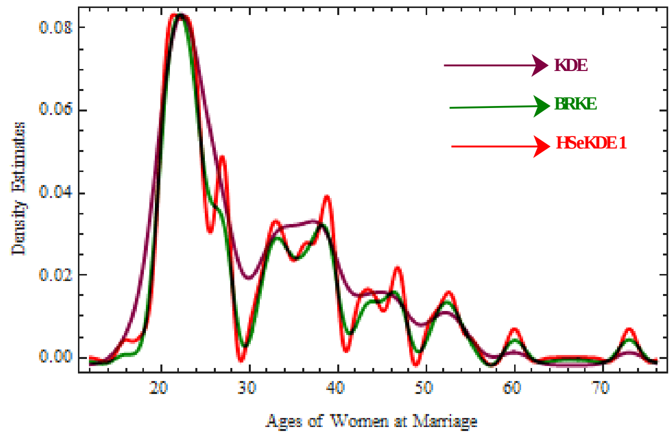

Fig. 8: Gaussian Kernel Estimates of Ages of Women at Marriage Time Data

The kernel estimates modality features of the investigation data as shown in Figures 5, 6, 7 and 8 with the Epanechnikov, Biweight, Triweight and Gaussian kernel estimates of the ages of men at marriage time by the estimators. A critical study of the Figure 8 being the kernel estimates of the data with the Gaussian kernel function and its smoothing parameters in this case as 1.63847, 1.58001and 1.14918 give the visualizations using the KDE, $BRKE$ and $HSeKDE$ 1 respectively. The multimodality property of the observed data was affirmed by the estimators with more visualized features displayed by the $BRKE$ and $HSeKDE$ 1. The Figure 8 revealed that at age 22 the region of the highest peak, indicates the probability of contracting more marriages was confirmed by the estimators. Table 2 gives smoothing parameter, Statistical qualities and measure of performance of these kernel estimators.

Table 2: ***Kernel Estimators Smoothing Parameters, Statistical Qualities and AMISE Values***

Estimators	Kernels	$R(K)$	$\mu_2(K)$	h_{opt}	$\int Bias^2(\tilde{m}_n(x))dx$	$\int Var(\tilde{m}_n(x))dx$	$AMISE$
		Ages at Marriage; Sample Size, $n = 100$.					
KDE	Epan.	0.60000	0.20000	2.88502	5.19927E-04	2.07971E-03	2.59964E-03
	Biweight	0.71429	0.14285	3.41778	5.18470E-04	2.08988E-03	2.60835E-03
	Triweight	0.81585	0.11111	3.88105	5.09532E-04	2.10213E-03	2.61166E-03
	Gaussian	$1/\sqrt{2\pi}$	1.00000	1.63847	1.35221E-03	5.40886E-03	6.76107E-03
$BRKE$	Epan.	0.60000	0.20000	3.45594	1.25875E-07	3.07791E-04	3.07917E-04
	Biweight	0.71429	0.14285	5.32563	7.53051E-08	7.91454E-04	7.91529E-04
	Triweight	0.81585	0.11111	6.21244	2.61142E-08	8.56140E-04	8.56166E-04
	Gaussian	$1/\sqrt{2\pi}$	1.00000	1.58001	1.74756E-04	1.39805E-03	1.57280E-03
$HSeKDE$ 1	Epan.	0.60000	0.20000	1.61681	6.69089E-09	1.09518E-07	1.16209E-07
	Biweight	0.71429	0.14285	1.77096	2.78794E-09	1.32760E-07	1.35548E-07
	Triweight	0.81585	0.11111	1.89598	1.45316E-09	1.45271E-07	1.45416E-07
	Gaussian	$1/\sqrt{2\pi}$	1.00000	1.14918	6.27626E-09	5.45915E-06	5.46543E-06

The comparison of the estimators error analysis in Table 2 reveals that as the power of the kernels increases, the smoothing parameter and $AMISE$ values are also on the increase with the $HSeKDE$ 1 having the minimum values of these parameters. The computed $AMISE$ values of the $HSeKDE$ 1 evidently shows it outperformed the other kernel estimators with the smallest value of error precision of 0.000000116209 and 0.00000546543 with the optimum and Gaussian kernels.

5. CONCLUSION

In this study, we conclude that the results obtained from the Hermite series kernel density estimator one are better that the kernel density estimator and bias reduced kernel estimator. We showed this by computing the asymptotic mean integrated squared error (AMISE) and the rates of convergence. The Table 1 and 2 showed the Statistical qualities and estimators performance parameters of the estimators, it becomes evidential that the Hermite series kernel density estimator one has the least asymptotic mean integrated squared error (AMISE) when compared with the other exiting estimators under real data applications.

REFERENCES

Delgado, M. A and Vidal-Sanz, J. N (2002): Average Singular Integral Estimation as a Bias Reduction Technique. Journal of Multivariate Analysis, 80 (1): 127-137. Doi:10.1006/jmva.2000.1978.

Ejakpovi, S. U and Ozobokeme, J. K (2024): A Nonparametric Model to the Distribution Dynamics of Maritime Piracy in Economic Growth in Nigeria. The Journal of the Mathematical Association of Nigeria, Ogun State (JOMANOGS), Volume 6, No. 2, 53-66.

Fleming, C.H and Calabrese, J.M (2017): A New kernel density estimator for accurate home-range and Species-range area estimation. Methods in Ecology and Evolution, 8, 571-579.

Hang, H., Steinwart, I., Feng, Y and Suykens, J.A.K (2018): Kernel Density Estimation for Dynamical Systems. Journal of Machine Learning Research, Volume 19, 1-49.

Humbert, P., Bars, B and Minvielle, L (2022): Robust kernel Density Estimation with Median-of-means principles. Proceedingd of the 39th International conference on machine learning. Baltimore, Maryland, USA, PMLR162.

Lawal, Y.B., Owolawi, P.A., Tu, C., Wyke, E.V and Ojo, J.S (2024): The Kernel Density Estimation Technique for Spatio-Temporal Distribution and Mapping of Rain height over South Africa: The Effects on Rain-induced Attenuation. Atmosphere, 15, 1354. https://doi.org/10.3390/atmos15111354.

McClave, J. T and Benson, P. G (1988): Statistics for Business and Economics. Macmillan, USA.

Mynbaev, K, and Martin-Filho, C. B (2010): Bias reduction in Kernel DensityEstimation via Lipschitz Condition. Journal of Nonparametric Statistics, 22, 219-235.

Mynbaev, K, Martin-Filho, C. B and Aipenova, A (2015): A Class of Nonparametric Density Derivative Estimators Based on Global Lipschitz Conditions.

Raykar, V. C., Duraiswami, R and Zhao, L.H (2015): Fast Computation of Kernel Estimation. Journal of Graphical Statistics, Volume 19 (1), 205-220

Raykar, V.C, Duraiswami, R and Zhao, L.H (2015): Fast Computation of Kernel Estimation. Journal of Computational and Graphical Statistics, Volume 19 (1), 205-220.

Rodchuen, M and Suwattee, P (2011): Probability Density Estimation using Two New Kernel Functions. Chiang Mai Journal of Science, Volume 38 (1), 1-12.

Sha, M and Xie, Y (2016): The Study of Different Types of kernel Density Estimators. 2nd International conference on Electronic Network and Computer Engineering.

Siloko, I. U., Siloko, E. A., Ikpotokin, O., Ishiekwene, C. C. and Afere, B. A. (2019a): On Asymptotic Mean Integrated Squared Error's Reduction Techniques in Kernel Density Estimation. International Journal of Computational and Theoretical Statistics, 6(1), 89–98.

Siloko, I.U., Ejakpovi, S.U., Ukhurebor, K. E., Siloko, E. A and Ishiekwene, C.C (2022): Comparison of Kernel Density Function in Detecting Effects of Daily Emission of Sulphur (IV) Oxide from an Industrial Plant. Ethiopia Journal of Science and Technology. 15 (1): 53-66

Silverman, B. W. (2018). Density Estimation for Statistics and Data Analysis. Routledge, New York.

Tosatto, S., Akrour, R and Peters, J (2021): An upper Bound of the Bias ofNadaraya-Watson kernel Regression under Lipschitz Assumption. Stats 2021, 4, 1-17, https://doi.org/10.3390/stats4010001.

Weglarczyk, S (2018): Kernel density estimation and its application. ITM Web of Conference 23, 00037 (2018). XLVIII Seminar of Applied Mathematics, https://doi.org/10.1051/itmconf/20182300037.

Xie, X and Wu, J (2014): Some Improvement on Convergence Rates of Kernel Density Estimators. Journal of Applied Mathematics, Volume 5, 1684-1696, http://dx.doi-org/10.4236/am.2014.511161.

Ziane, Y., Zougab, N and Adjabi, S (2021): Body Tail Adaptive Kernel Estimation for nonnegative heavy-tailed data. Monte Carlo Methods and Applications. Volume 27 (1), 2021-2082. https://doi.org/10.1515/mcma.